AN OPTIMAL ALTERNATIVE THEOREM AND APPLICATIONS TO MATHEMATICAL PROGRAMMING*

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Abstract

Given a closed convex cone P with nonempty interior in a locally convex vector space, and a set $A \subset Y$, we provide various equivalences to the implication

$$A \cap (-\text{int } P) = \emptyset \Longrightarrow \text{co}(A) \cap (-\text{int } P) = \emptyset,$$

among them, to the pointedness of cone$(A+\text{int } P)$. This allows us to establish an optimal alternative theorem, suitable for vector optimization problems. In addition, we present an optimal alternative theorem which characterizes two-dimensional spaces in the sense that it is valid if, and only if, the space is at most two-dimensional. Applications to characterizing weakly efficient solutions through scalarization; the zero (Lagrangian) duality gap; the Fritz-John optimality conditions for a class of nonconvex nonsmooth minimization problems, are also presented.

Key Words. Theorem of the alternative, vector optimization, generalized subconvexlike set, weakly efficient solution.

1 Introduction and formulation of the problem

Alternative theorems are very useful to derive many important results in convex and nonconvex optimization theory: the existence of Lagrange multipliers, duality results, scalarization of vector functions, etc. To be precise, let us consider a real locally convex topological vector space Y and a closed convex cone $P \subseteq Y$ such that int $P \neq \emptyset$. We denote by Y^* the topological dual space of Y, and by P^* the (positive) polar cone of P. Given a nonempty set $A \subseteq Y$, alternative theorems assert the validity of exactly one of the following assertions:

$$\exists\ a \in A \text{ such that } a \in -\text{int } P; \tag{1}$$

$$\exists\ p^* \in P^*,\ p^* \neq 0, \text{ such that } \langle p^*, a\rangle \geq 0 \ \ \forall\ a \in A. \tag{2}$$

Here $\langle \cdot, \cdot \rangle$ stands for the duality pairing between Y and Y^* and int P denotes the topological interior of P. We recall that P^* is defined by

$$P^* = \{p^* \in Y^* : \ \langle p^*, p\rangle \geq 0 \ \ \forall\ p \in P\}.$$

The closedness and convexity of the cone P is equivalent to $P = P^{**}$ by the bipolar theorem. In this case,

$$p \in P \iff \langle p^*, p\rangle \geq 0 \ \ \forall\ p^* \in P^*.$$

Moreover,

$$p \in \text{int } P \iff \langle p^*, p\rangle > 0 \ \ \forall\ p^* \in P^* \setminus \{0\}. \tag{3}$$

A separation theorem for convex sets and the above remarks allow us to write (1) and (2) in an equivalent way as, respectively,

$$A \cap (-\text{int } P) \neq \emptyset, \tag{4}$$

$$\text{co}(A) \cap (-\text{int } P) = \emptyset, \tag{5}$$

where "co(A)" stands for the convex hull of A. While the inconsistency of both assertions (4) and (5) is straightforward, the proof of the implication

$$A \cap (-\text{int } P) = \emptyset \Longrightarrow \text{co}(A) \cap (-\text{int } P) = \emptyset, \tag{6}$$

requires a careful analysis due to the lack of convexity of A. One of the goals of the present paper is to characterize those sets A for which implication (6) is true. Most papers appearing in the literature (see for instance [1, 10, 13, 19, 20] and the references therein) were concerned with providing some (sufficient) conditions implying (6). In this spirit various generalizations of the usual notion of convexity were introduced. Some of them will be discussed in Section 3.

Several of our results can be derived for cones P with nonempty quasi-interior, thus allowing the (topological) interior to be empty. In Section 2 we give the necessary definitions, together with some elementary results about cones. In Section 3 we show that (5) can be restated in terms of pointedness of the set cone($A + \text{int } P$). At the same time, we compare several of the previously introduced notions of generalized convexity for sets and vector valued functions, and show equivalences between them. As a consequence of these results, we are able to derive and strenghten several of the already known alternative theorems. In Section 4 we provide a complete characterization of those sets A in $\mathbb{R}^2$ for which (6) holds, and show that this characterization holds if and only if the space is at most 2-dimensional.

As an illustrative application of our main result, we characterize in Section 5 those mappings $F : K \to \mathbb{R}^2$ for which the equivalence

$$\bar{x} \in E_w \Longleftrightarrow \bar{x} \in \bigcup_{p^* \in P^*, p^* \neq 0} \text{argmin}_K \langle p^*, F(\cdot) \rangle$$

holds, where E_w denotes the set of weakly efficient solutions to F on K (see Section 5). Such an equivalence was crucial to develop a well-posedness theory in vector optimization in [5]. Quadratic scalarization instead of linear was employed in [6] to compute efficient solutions.

Other applications concern the zero (Lagrangian) duality gap, and the Fritz-John optimality conditions for a class of nonconvex minimization problems without smoothness.

2 Some basic notation and preliminaries

Throughout the paper, X will be a vector space and Y a real locally convex topological vector space. We will denote by $\langle \cdot, \cdot \rangle$ the duality pairing between Y and Y^*. Given $x, y \in X$ we set $[x, y] = \{tx + (1-t)y : t \in [0,1]\}$. The segments $]x, y]$ etc are defined analogously.

By cone we mean a set $P \subseteq Y$ such that $tP \subseteq P \ \forall \ t \geq 0$; given $A \subseteq Y$, cone(A) stands for the smallest cone containing A, that is,

$$\text{cone}(A) = \bigcup_{t \geq 0} tA,$$

whereas $\overline{\text{cone}}(A)$ denotes the smallest closed cone containing A: obviously $\overline{\text{cone}}(A) = \overline{\text{cone}(A)}$, where $\overline{A}$ denotes the closure of A. Furthermore, we put

$$\text{cone}_+(A) \doteq \bigcup_{t>0} tA.$$

Evidently, $\text{cone}(A) = \text{cone}_+(A) \cup \{0\}$, and therefore, $\overline{\text{cone}(A)} = \overline{\text{cone}_+(A)}$. In [13, 19, 20] the notation cone(A) instead of cone$_+(A)$ is employed.

Given a convex subset K of Y, an element $x \in K$ is called a quasi-interior point if there is no closed hyperplane supporting K at x; i.e., if for all $x^* \in Y^*$ the following implication holds:

$$\langle x^*, y \rangle \geq \langle x^*, x \rangle \text{ for all } y \in K \ \Rightarrow \ x^* = 0.$$

Equivalently, x is an quasi-interior point if and only if $\overline{\text{cone}}(K-x) = Y$ (see for instance [3] for details and references on quasi-interiors). We will denote by qint K the set of quasi-interior points of K. If int $K \neq \emptyset$, then int $K =$ qint K. For this reason, all results in this paper involving qint K are also true for int K, provided the latter set is nonempty. On the other hand, for any $p \in [1, +\infty)$ the positive cone $l^p_+ = \{(x_i)_{i \in \mathbb{N}} \in l^p : x_i \geq 0, \ \forall i \in \mathbb{N}\}$ of the space $l^p = \left\{(x_i)_{i\in\mathbb{N}} : \sum_{i \in \mathbb{N}} |x_i|^p < +\infty\right\}$ has nonempty quasi-interior, but its interior (and even the relative algebraic interior) is empty. Quasi-interior points share some properties of the interior points; for instance, if $x \in$ qint K and $y \in K$ then $[x, y[\ \subseteq$ qint K. In particular, qint K is convex and dense in K whenever it is not empty.

If P is a closed convex cone, then it is easy to check that $x \in$ qint P if and only if $\langle x^*, x \rangle > 0$ for all $x^* \in P^* \backslash \{0\}$, or equivalently if the set $B = \{x^* \in P^* : \langle x^*, x \rangle = 1\}$ is a w*-closed base for P^* (we recall that a convex set B is called a base for P^* if 0 is not in the w*-closed hull of B and $P^* = \text{cone}(B)$). If $P \neq Y$, then $0 \notin$ qint P. Note also that qint $P = \text{cone}_+(\text{qint } P)$ and $P +$ qint $P =$ qint P.

Assumption *In the rest of the paper, $P \subseteq Y$ will be a closed convex cone with $P \neq Y$ and* qint $P \neq \emptyset$.

Some elementary properties of sets to be used later are collected in the next proposition.

Proposition 2.1 *Let $A \subseteq Y$ be any nonempty set.*

(a) $\alpha A + (1-\alpha)A \subseteq \mathrm{cone}(A)\ \forall\, \alpha \in\ ]0,1[\ \Longleftrightarrow \mathrm{cone}(A)$ *is convex* $\Longleftrightarrow \mathrm{co}(A) \subseteq \mathrm{cone}(A)$.

(b) $\alpha A + (1-\alpha)A \subseteq \mathrm{cone}_+(A)\ \forall\, \alpha \in\ ]0,1[\ \Longleftrightarrow \mathrm{cone}_+(A)$ *is convex* $\Longleftrightarrow \mathrm{co}(A) \subseteq \mathrm{cone}_+(A)$.

(c) $\mathrm{cone}_+(A+M) = \mathrm{cone}_+(A) + M$ *provided that* M *is such that* $tM \subseteq M\ \forall\, t > 0$.

(d) $\mathrm{cone}(A) + M \subseteq \overline{\mathrm{cone}}(A+M)$ *and* $\overline{\mathrm{cone}(A)+M} = \overline{\mathrm{cone}}(A+M)$, *provided that* M *is a cone.*

(e) $\overline{\mathrm{cone}}(A + \mathrm{qint}\ P) = \overline{\mathrm{cone}}(A+P)$, *provided that* P *is a convex cone with* $\mathrm{qint}\ P \neq \emptyset$.

(f) $\mathrm{cone}_+(A + \mathrm{int}\ P)$ *is convex* $\Longleftrightarrow \mathrm{cone}(A + \mathrm{int}\ P)$ *is convex* $\Longleftrightarrow \overline{\mathrm{cone}}(A+P)$ *is convex, provided that* P *is a convex cone with* $\mathrm{int}\ P \neq \emptyset$.

Proof. The proof of (a), (b) and (c) is straightforward.
(d): According to (c), $\mathrm{cone}_+(A) + M = \mathrm{cone}_+(A+M) \subseteq \overline{\mathrm{cone}}(A+M)$. On the other hand, for a fixed $a \in A$, every $p \in M$ can be obtained as the limit of $\frac{1}{n}(a+np)$. Hence $M \subseteq \overline{\mathrm{cone}}(A+M)$ and this shows the inclusion in (d). Since obviously $\mathrm{cone}(A+M) \subseteq \mathrm{cone}(A) + M$, the equality of closures also follows.
(e): Since $\mathrm{qint}\ P \subseteq P$, we have $\overline{\mathrm{cone}}(A + \mathrm{qint}\ P) \subseteq \overline{\mathrm{cone}}(A+P)$. Also, from $P \subseteq \overline{\mathrm{qint}\ P}$ it follows that $A + P \subseteq A + \overline{\mathrm{qint}\ P} \subseteq \overline{A + \mathrm{qint}\ P} \subseteq \overline{\mathrm{cone}}(A + \mathrm{qint}\ P)$, hence ($e$) follows.
(f): If $\mathrm{cone}_+(A + \mathrm{int}\ P)$ is convex, then it easily follows that $\mathrm{cone}(A + \mathrm{int}\ P)$ is convex. By using (e), we deduce that $\overline{\mathrm{cone}}(A+P)$ is convex. If $\overline{\mathrm{cone}}(A+P)$ is convex, then $\mathrm{cone}_+(A + \mathrm{int}\ P)$ is convex by Theorem 2.6 in [14]. □

Remark 2.2 Proposition 2.1(f) does not hold with $\mathrm{qint}\ P$ in the place of $\mathrm{int}\ P$. Indeed, let $Y = l^1$ and $P = l^1_+$. Then $\mathrm{qint}\ l^1_+ = \left\{(\alpha_i)_{i\in\mathbb{N}} : \alpha_i > 0\right\}$ while $\mathrm{int}\ l^1_+ = \emptyset$. Set

$$A = l^1 \backslash \left(-\mathrm{qint}\ l^1_+\right) = \left\{(\alpha_i)_{i\in\mathbb{N}} : \exists\ i \in \mathbb{N} \text{ with } \alpha_i \geq 0\right\}.$$

Each $(a_i)_{i\in\mathbb{N}} \in l^1$ can be written as a limit of a sequence of elements each of which has a finite number of nonzero coordinates. Thus $\overline{A} = l^1$ and $\overline{\mathrm{cone}}(A + l^1_+) = l^1$ is convex. However, one can readily check that $\mathrm{cone}_+(A + \mathrm{qint}\ P) = A + \mathrm{qint}\ P = \left\{(\alpha_i)_{i\in\mathbb{N}} : \exists\ i \in \mathbb{N} \text{ with } \alpha_i > 0\right\}$ is not convex.

3 The alternative theorem in spaces of arbitrary dimension

In search of conditions implying the validity of (6), several relaxed notions of convexity have appeared in the literature. Before reviewing and comparing some of them, we will first reformulate the conclusion of the alternative theorem in terms of the cone cone $(A + \text{qint } P)$. We recall the definition of pointedness for a cone that is not necessarily convex (see for instance [12]).

Definition 3.1 *A cone $K \subseteq Y$ is called "pointed" if $x_1 + \ldots + x_k = 0$ is impossible for $x_1, x_2, \ldots, x_k$ in K unless $x_1 = x_2 = \ldots = x_k = 0$.*

Our first result is the following:

Theorem 3.2 *Let $A \subseteq Y$ be any nonempty set and $P \subseteq Y$, $P \neq Y$, be a convex and closed cone such that* $\text{qint } P \neq \emptyset$. *The following assertions are equivalent:*

- (a) $\text{cone}(A + \text{qint } P)$ *is pointed;*
- (b) $\text{co}(A) \cap (-\text{qint } P) = \emptyset$.

Proof. We first prove

$$\text{cone}(A + \text{qint } P) \text{ is pointed} \implies A \cap (-\text{qint } P) = \emptyset. \tag{7}$$

If there exists $x \in A \cap (-\text{qint } P)$, then $x = 2(x - \frac{x}{2}) \in \text{cone}(A + \text{qint } P)$ and $-x = x + (-2x) \in A + \text{qint } P \subseteq \text{cone}(A + \text{qint } P)$. By pointedness, $x = 0$, hence $0 \in \text{qint } P$. As noted in Section 2, this implies $P = Y$, a contradiction.

Now assume that (a) holds. If (b) does not hold, then there exists $x \in -\text{qint } P$ such that $x = \sum_{i=1}^{m} \lambda_i a_i$ with $\sum_{i=1}^{m} \lambda_i = 1$, $\lambda_i > 0$, $a_i \in A$. Thus, $0 = \sum_{i=1}^{m} \lambda_i (a_i - x)$. Using (a) we infer that $\lambda_i(a_i - x) = 0$ for all $i = 1, \ldots, m$. This contradicts (7).

Conversely, assume that (b) holds. If $\text{cone}(A + \text{qint } P)$ is not pointed, then there exist $x_i \in \text{cone}(A + \text{qint } P)\backslash\{0\}$, $i = 1, 2, \ldots n$, such that $\sum_{i=1}^{n} x_i = 0$. Each x_i can be written as $x_i = \lambda_i(y_i + u_i)$ with $\lambda_i > 0$, $y_i \in A$ and $u_i \in \text{qint } P$. Hence $\sum_{i=1}^{n} \lambda_i y_i = -\sum_{i=1}^{n} \lambda_i u_i$. Setting $\mu_i = \lambda_i / \sum_{j=1}^{n} \lambda_j$ we get $\sum_{i=1}^{n} \mu_i y_i = -\sum_{i=1}^{n} \mu_i u_i \in \text{co}(A) \cap (-\text{qint } P)$, a contradiction. $\square$

When $\text{int } P \neq \emptyset$, then by the separation theorem $\text{co}(A) \cap (-\text{qint } P) = \emptyset$ is equivalent to the existence of $p^* \in P^* \backslash \{0\}$ such that $\langle p^*, y \rangle \geq 0$ for all $y \in A$. Thus, in case the set A is the image of some vector-valued mapping, the previous theorem implies the following

Corollary 3.3 *Let $K \subseteq X$ be any nonempty set, $P \subseteq Y$ be a closed convex cone such that* $\text{int } P \neq \emptyset$, *and $G : K \to Y$ be any mapping. Then the following assertions are equivalent:*

(a) $\text{cone}(G(K) + \text{int } P)$ *is pointed;*

(b) $\exists\, p^* \in P^*,\ p^* \neq 0,\ \langle p^*, G(x)\rangle \geq 0\ \forall\, x \in K.$

We now recall the most general among the relaxed notions of convexity that were used in alternative theorems.

Definition 3.4 *Let $P \subseteq Y$ be a closed convex cone with nonempty interior. A set $A \subseteq Y$ is called:*

(a) *generalized subconvexlike [20] if* $\exists\, u \in \text{int } P,\ \forall\, x_1,\ x_2 \in A,\ \forall\, \alpha \in\ ]0,1[,\ \forall\, \varepsilon > 0,$ $\exists\, \rho > 0$ *such that*

$$\varepsilon u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P; \tag{8}$$

(b) *presubconvexlike if* $\exists\, u \in Y,\ \forall\, x_1,\ x_2 \in A,\ \forall\, \alpha \in\ ]0,1[,\ \forall\, \varepsilon > 0,\ \exists\, \rho > 0$ *such that (8) holds;*

(c) *nearly subconvexlike [13, 19] if* $\overline{\text{cone}}(A + P)$ *is convex.*

Note that the definition of presubconvexlike sets is a transcription of an analogous definition for Y-valued functions given in [21]. Also, from Proposition 2.1(f) it follows that (c) above is equivalent to the convexity of $\text{cone}_+(A + \text{int } P)$ and also to the convexity of $\text{cone}(A + \text{int } P)$. In fact, we will show that all three notions of generalized convexity of sets given in Definition 3.4 are equivalent.

Proposition 3.5 *In Definition 3.4,* (a), (b) *and* (c) *are equivalent.*

Proof. (a) $\Leftrightarrow$ (b): It is obvious that (a) implies (b). If A is presubconvexlike, let $u \in Y$ be the element whose existence is required by (b). Since $\text{int } P - \text{int } P = Y$ (see, cg., [11]) we can write $u = v - w$ with $v, w \in \text{int } P$. By assumption, for every $x_1,\ x_2 \in A,\ \alpha \in\ ]0,1[,\ \varepsilon > 0$ there exists $\rho > 0$ such that (8) holds. Then

$$\varepsilon v + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P + \varepsilon w \subseteq \rho A + P.$$

Thus, A is generalized subconvexlike.

(a) $\Rightarrow$ (c): In Theorem 2.1 of [20], it is shown that a generalized subconvexlike set A is such that the set $\text{cone}_+(A) + \text{int } P$ is convex. By Proposition 2.1(c)(f), $\overline{\text{cone}}(A + P)$ is convex.

(c) $\Rightarrow$ (a): If $\overline{\text{cone}}(A + P)$ is convex then by Proposition 2.1(f), $\text{cone}_+(A + \text{int } P)$ is convex. From (b) of the same proposition applied to the set $A + \text{int } P$ it follows that

$$\alpha A + (1-\alpha)A + \text{int } P \subseteq \text{cone}_+(A + \text{int } P)\ \ \forall\, \alpha \in\ ]0,1[.$$

This allows us to conclude that A is generalized subconvexlike. $\square$

Thus, the two alternative theorems in [19] and [20] (with "int" instead of "qint") can be unified and extended as follows:

Theorem 3.6 *Let $A \subseteq Y$ be any nonempty set. Assume that $A \cap (-\mathrm{qint}\ P) = \emptyset$. Then*

$$\mathrm{cone}_+(A + \mathrm{qint}\ P) \text{ is convex } \Longrightarrow \mathrm{co}(A) \cap (-\mathrm{qint}\ P) = \emptyset.$$

It is now clear that Theorem 3.6 is a consequence of Theorem 3.2 and the following easy proposition:

Proposition 3.7 *If* $\mathrm{cone}_+(A + \mathrm{qint}\ P)$ *is convex and* $A \cap (-\mathrm{qint}\ P) = \emptyset$*, then* $\mathrm{cone}(A + \mathrm{qint}\ P)$ *is pointed.*

Proof. Since $\mathrm{cone}(A + \mathrm{qint}\ P)$ is also a convex cone, we have to show that whenever $x, -x \in \mathrm{cone}(A + \mathrm{qint}\ P)$, then $x = 0$. Indeed, assume that $x \neq 0$. Then $x, -x \in \mathrm{cone}_+(A + \mathrm{qint}\ P)$. This last set is convex, hence $0 = x + (-x) \in \mathrm{cone}_+(A + \mathrm{qint}\ P)$. Thus, there exist $\lambda > 0$, $y \in A$ and $u \in \mathrm{qint}\ P$ such that $0 = \lambda(y + u)$. Then $y \in A \cap (-\mathrm{qint}\ P)$, a contradiction. $\square$

The converse of Proposition 3.7 (or Theorem 3.6) does not hold, as shown by the following example.

Example 3.8 Let us consider in $\mathbb{R}^3$ the polyhedral (closed convex) cone $P = \mathrm{cone}(B)$, where

$$B = \Big\{(1, -x_2, x_3) :\ 0 \leq x_2, 0 \leq x_3, x_2 + x_3 \leq 1\Big\},$$

and the set

$$A = \Big\{(x_1, 1, \sqrt{1 - x_1^2}) :\ 0 \leq x_1 \leq 1\Big\}.$$

It is not difficult to check that $\mathrm{co}(A) \cap (-\mathrm{int}\ P) = \emptyset$ thus $\mathrm{cone}(A + \mathrm{int}\ P)$ is pointed. However, we will see that $\overline{\mathrm{cone}}(A + P)$ is nonconvex. To this end, it is enough to show that $z = (\frac{1}{2}, 1, \frac{1}{2}) \notin \overline{\mathrm{cone}}(A + P)$ since $z = \frac{1}{2}x + \frac{1}{2}y$ with $x = (0, 1, 1) \in A$ and $y = (1, 1, 0) \in A$. Assume on the contrary that there exist sequences $0 \leq x_1^k \leq 1$, $0 \leq x_2^k \leq 1$, $0 \leq x_3^k \leq 1$ and β_k, $\lambda_k \geq 0$ such that

$$\lambda_k(x_1^k + \beta_k) \to \frac{1}{2}, \tag{9}$$

$$\lambda_k(1 - \beta_k x_2^k) \to 1, \tag{10}$$

$$\lambda_k(\sqrt{1 - (x_1^k)^2} + \beta_k x_3^k) \to \frac{1}{2}. \tag{11}$$

If λ_k is bounded, we may assume that $\lambda_k \to \lambda$ for some $\lambda \geq 0$. From (10), we obtain $\lambda \geq 1$. On the other hand, up to a subsequence $x_1^k \to x_1$, thus (9) implies $x_1 \leq \frac{1}{2}$. By (11) we get $\sqrt{1 - x_1^2} \leq \frac{1}{2}$, which in turn gives $x_1 \geq \frac{\sqrt{3}}{2}$, contradicting a previous inequality. We now assume that $\lambda_k \to +\infty$. From (9) it follows $x_1^k \to 0$. Taking k sufficiently large, (11) yields a contradiction.

The preceding definitions of relaxed convexity for sets induce corresponding definitions for vector valued mappings: given a nonempty subset K of X, a multivalued mapping $G : K \rightrightarrows Y$ is called generalized subconvexlike [20] (respectively, nearly subconvexlike [13, 19], presubconvexlike [21]) if the set $G(K)$ is generalized subconvexlike (resp., nearly subconvexlike, presubconvexlike) . According to Proposition 3.5, these three notions are identical. Other definitions of generalized convexity for (single-valued) vector valued functions in view of using them to alternative theorems were given in [10] and [16]. A mapping $G : K \to Y$ is called $*$-quasiconvex [10] if $\langle x^*, G(\cdot) \rangle$ is quasiconvex for all $x^* \in P^*$. It is called naturally-P-quasiconvex [16] if for all $x, y \in K$, $G([x, y]) \subseteq [G(x), G(y)] - P$. We will first show that these notions are equivalent:

Proposition 3.9 *Let $K \subseteq X$ be any nonempty convex set and $P \subseteq Y$ be a closed convex cone with nonempty interior. Then a mapping $G : K \to Y$ is $*$-quasiconvex if and only if it is naturally-P-quasiconvex.*

Proof. Assume that G is naturally-P-quasiconvex. We need to check that given $t \in \mathbb{R}$ and $x^* \in P^*$, the set $K_t = \{z \in K : \langle x^*, G(z) \rangle \leq t\}$ is convex. Indeed, if $x, y \in K_t$ then by natural-P-quasiconvexity of G, for all $z \in [x, y]$ there exists $\lambda \in [0, 1]$ and $u \in P$ such that $G(z) = \lambda G(x) + (1 - \lambda)G(y) - u$. Hence,

$$\langle x^*, G(z) \rangle = \lambda \langle x^*, G(x) \rangle + (1 - \lambda) \langle x^*, G(y) \rangle - \langle x^*, u \rangle \leq t$$

thus $z \in K_t$, so K_t is convex.

Conversely, assume that G is not naturally-P-quasiconvex. Then there exist $x, y \in K$ and $z \in\ ]x, y[$ such that for all $\mu \in [0, 1]$, $G(z) \notin \mu G(x) + (1 - \mu)G(y) - P$. Thus for every $\mu \in [0, 1]$ there exists $x^* \in Y^* \setminus \{0\}$ such that

$$\langle x^*, G(z) \rangle > \langle x^*, \mu G(x) + (1 - \mu)G(y) - u \rangle \ \ \forall\ u \in P.$$

Since P is a cone, we get $\langle x^*, u \rangle \geq 0$ for all $u \in P$, i.e., $x^* \in P^*$, and also $\langle x^*, G(z) - \mu G(x) - (1 - \mu)G(y) \rangle > 0$. Since by assumption int $P \neq \emptyset$, there exists a w^*-compact base B of P^*. Setting $f(y^*, \mu) = \langle y^*, G(z) - \mu G(x) - (1 - \mu)G(y) \rangle$ we get

$$\max_{y^* \in B} \min_{\mu \in [0,1]} f(y^*, \mu) = \min_{\mu \in [0,1]} \max_{y^* \in B} f(y^*, \mu) > 0.$$

Hence there exists $x^* \in B$ such that

$$\langle x^*, G(z)\rangle > \mu\langle x^*, G(x)\rangle + (1-\mu)\langle x^*, G(y)\rangle \;\; \forall\, \mu \in [0,1].$$

In particular, we get $\langle x^*, G(z)\rangle > \langle x^*, G(x)\rangle$ and $\langle x^*, G(z)\rangle > \langle x^*, G(y)\rangle$. Thus G is not $*$-quasiconvex. $\square$

In [10] it is proven that implication (6) holds for $A = G(K)$ under the $*$-quasiconvexity of G and the assumption

$$\forall p^* \in P^*, \text{ the restriction of } \langle p^*, G(\cdot)\rangle \text{ on any line segment of } K \text{ is lower semicontinuous.} \tag{12}$$

We will see that the $*$-quasiconvexity of G together with (12) imply the convexity of cone$(G(K) + \text{int } P)$ thus, in particular, that G is nearly subconvexlike. This follows from the next proposition which is of interest by itself. We refer the reader to [9] for the definition of upper semicontinuity and other properties of multivalued mappings that will be used in the proof.

Proposition 3.10 *Let $K \subseteq X$ be any nonempty convex set, $P \subseteq Y$ be a closed convex cone and $G : K \to Y$ be naturally-P-quasiconvex and satisfying (12). Then*

$$\forall\; x, y \in K, \;\; [G(x), G(y)] \subseteq G([x,y]) + P. \tag{13}$$

Proof. Given $x, y \in K$, define $H : [x,y] \rightrightarrows [G(x), G(y)]$ by $H(z) = (G(z) + P) \cap ([G(x), G(y)])$. We show first that H is closed. Let (z_n, w_n), $n \in \mathbb{N}$, be a sequence in the graph of H, converging to (z, w). Then $w_n \in H(z_n) \subseteq [G(x), G(y)]$. Obviously, $w \in [G(x), G(y)]$. Also, for every $n \in \mathbb{N}$ there exists $v_n \in P$ such that $w_n = G(z_n) + v_n$. For each $p^* \in P^*$ we get by assumption (12):

$$\begin{aligned}\langle p^*, w - G(z)\rangle &\geq \lim \langle p^*, w_n\rangle - \liminf \langle p^*, G(z_n)\rangle \\ &= \lim \langle p^*, w_n\rangle + \limsup \langle p^*, -G(z_n)\rangle \\ &= \limsup \langle p^*, v_n\rangle \geq 0.\end{aligned}$$

Since this is true for all $p^* \in P^*$, we deduce that $w - G(z) \in P$, i.e., $w \in H(z)$ and H is closed. Hence, H is upper semicontinuous.

Also, for every $z \in [x,y]$, $H(z) \neq \emptyset$ by the definition of natural-P-quasiconvexity. In addition, $H(z)$ is connected, being convex. Hence, the image of $[x,y]$ through H is connected (cf. Proposition 2.24, pg. 43 in [9]). This image is a subset of the line segment $[G(x), G(y)]$. Since $G(x) \in H(x)$ and $G(y) \in H(y)$, we deduce that $H([x,y]) = [G(x), G(y)]$. Thus, for every $w \in [G(x), G(y)]$ there exists $z \in [x,y]$ such that $w \in H(z)$, i.e., $w = G(z) + u$ for some $u \in P$. This shows inclusion (13). $\square$

We deduce the following:

Corollary 3.11 *Let X, Y, P, G be as in the previous proposition. Then $G(K) + P$ is convex.*

Proof. It is sufficient to show that whenever $t \in [0,1]$, $x, y \in K$ and $u \in P$ then $tG(x) + (1-t)G(y) + u \in G(K) + P$. But this is obvious in view of the proposition. $\square$

Thus, given a cone P with int $P \neq \emptyset$, if a mapping G is $*$-quasiconvex (or, equivalently, naturally-P-quasiconvex) and satisfies (12), then $G(K) + P$ is convex. This implies that G is nearly subconvexlike, so the alternative theorems of [10] and [16] are included in Theorem 3.6 and in particular in Theorem 3.2. The converse does not hold: the mapping $G(x) = (x, f(x))$, $x \in [-1,1]$, where $f(x) = 1 - |x|$, is clearly nearly subconvexlike (with $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$), but the real-valued function $x \in [-1,1] \mapsto \langle (0,1), (x, f(x)) \rangle = f(x)$ is not quasiconvex, that is, G is not $*$-quasiconvex.

4 Characterizing the two-dimensionality through the alternative theorem

According to Theorem 3.6 (see also Proposition 2.1(f)), whenever $A \cap (-\text{int } P) = \emptyset$ holds, the convexity of $\text{cone}(A + \text{int } P)$ is a sufficient condition for $\text{co}(A) \cap (-\text{int } P) = \emptyset$ to hold. We will now see that in case $Y = \mathbb{R}^2$, it is also necessary.

Theorem 4.1 *Let $P \subseteq \mathbb{R}^2$ be a convex closed cone such that* int $P \neq \emptyset$, *and $A \subseteq \mathbb{R}^2$ be any nonempty set satisfying $A \cap (-\text{int } P) = \emptyset$. Then the following assertions are equivalent:*

(a) $\text{co}(A) \cap (-\text{int } P) = \emptyset$;

(b) $\text{cone}(A + P)$ *is convex;*

(c) $\text{cone}(A + \text{int } P)$ *is convex;*

(d) $\text{cone}(A) + P$ *is convex;*

(e) $\overline{\text{cone}}(A + P)$ *is convex.*

Proof. According to Proposition 2.1(f), $(c) \Longleftrightarrow (e)$. It is obvious that (b) implies (e). Also, (d) implies (e) since $\overline{\text{cone}}(A + P)$ is the closure of $\text{cone}(A) + P$ (see Proposition 2.1(e)). Now assume that (c) holds. Then (e) holds. Due to the two-dimensionality of the space, the convex cone $\text{cone}(A + \text{int}\, P)$, being generated by the open set $A + \text{int}\, P$, is the open cone contained between two half lines, together with the origin; its closure $\overline{\text{cone}}(A+P)$ is the union of the former set and the half lines. Note that

$$\text{cone}(A + \text{int } P) \subseteq \text{cone}(A+P) \subseteq \text{cone}(A) + P \subseteq \overline{\text{cone}}(A+P) \tag{14}$$

where the last inclusion follows from Proposition 2.1(d). Thus, each of the cones $\text{cone}(A)+P$, $\text{cone}(A+P)$ appearing in (14) can be either $\text{cone}(A+\text{int}\,P)$, or its union with one of the halflines, or $\overline{\text{cone}}(A+P)$; in all cases, $\text{cone}(A)+P$, $\text{cone}(A+P)$ are convex, thus (b) and (d) hold. Consequently, (b), (c), (d) and (e) are equivalent.

That (e) implies (a) follows from Theorem 3.6 and Proposition 2.1(f).
(a) $\Rightarrow$ (b): There exists $x^* \in \mathbb{R}^2$ such that $\langle x^*, x\rangle \geq \langle x^*, u\rangle$ for all $x \in A$ and $u \in -\text{int } P$. It follows that $x^* \in P^*$ and $\langle x^*, x\rangle \geq 0$ for all $x \in A$, thus also for all $x \in \text{cone}(A+P)$.

Choose $u \in \text{int } P$. Let $y, z \in A$. Then obviously

$$\text{cone}(\{y\}) + \text{cone}(\{u\}) = \{\lambda y + \mu u : \lambda, \mu \geq 0\}$$

is a closed convex cone containing y and u and contained in $\text{cone}(A+P)$. The same is true for the cone $\text{cone}(\{z\}) + \text{cone}(\{u\})$. The two cones have the line $\text{cone}(\{u\})$ in common and their union is contained in $\text{cone}(A+P)$, thus it is contained in the halfspace $\{x \in \mathbb{R}^2 : \langle x^*, x\rangle \geq 0\}$. Hence, the set $B \doteq (\text{cone}(\{y\}) + \text{cone}(\{u\})) \cup (\text{cone}(\{z\}) + \text{cone}(\{u\}))$ is a convex cone. Since $y, z \in B$ we deduce that $[y, z] \subseteq B \subseteq \text{cone}(A+P)$ thus $\text{co}(A) \subseteq \text{co}(B) = B \subseteq \text{cone}(A+P)$. We deduce that $\text{cone}(A+P)$ is convex. $\square$

We now show that the equivalence between (a) and one of (b), (c), (d), (e) in Theorem 4.1 is characteristic of 2–dimensional spaces. Since, say, (b) $\Rightarrow$ (a) is a consequence of Theorem 3.6, we only consider the implication (a) $\Rightarrow$ (b) etc.

Theorem 4.2 *Let Y be a locally convex topological vector space and $P \subseteq Y$ be a closed, convex cone such that* $\text{int } P \neq \emptyset$ *and* $\text{int } P^* \neq \emptyset$*. The following assertions are equivalent:*

(a) *for all sets $A \subseteq Y$ one has*

$$\text{co}(A) \cap (-\text{int } P) = \emptyset \Rightarrow \overline{\text{cone}}(A+P) \ \text{ is convex;}$$

(b) *for all sets $A \subseteq Y$ one has*

$$\text{co}(A) \cap (-\text{int } P) = \emptyset \Rightarrow \text{cone}(A) + P \ \text{ is convex;}$$

(c) *for all sets $A \subseteq Y$ one has*

$$\text{co}(A) \cap (-\text{int } P) = \emptyset \Rightarrow \text{cone}(A + \text{int } P) \ \text{ is convex;}$$

(d) *Y is at most two-dimensional.*

Proof. We show first that (a) implies (d). Assume that the dimension of Y is at least 3. Let $x^* \in \text{int } P^*$. Then for all $x \in P\backslash\{0\}$, $\langle x^*, x\rangle > 0$. Fix $x \in \text{int } P$, and choose linearly independent $y, z \in Y$ such that $\langle x^*, y\rangle = \langle x^*, z\rangle = 0$ (this is possible since the dimension of the kernel of x^* is at least 2). In particular, y and z are not zero. Let A be the set $[y+z, y+x] \cup [y+x, y-z]$. Every element w of A has the form: $w = t\,(y \pm z) + (1-t)\,(y+x)$ with $t \in [0,1]$. Hence $\langle x^*, w\rangle = (1-t)\,\langle x^*, x\rangle \geq 0$. It follows that for every $w \in \text{co}(A)$, $\langle x^*, w\rangle \geq 0$. Since for every $u \in -\text{int } P$, $\langle x^*, u\rangle < 0$, it follows that $\text{co}(A) \cap (-\text{int } P) = \emptyset$.

We now show that $\overline{\text{cone}}(A+P)$ is not convex. Since $y = \frac{y+z}{2} + \frac{y-z}{2} \in \text{co}(A) \subseteq \text{co}\,(\overline{\text{cone}}\,(A+P))$, it is sufficient to show that $y \notin \overline{\text{cone}}\,(A+P)$. Suppose to the contrary that $y \in \overline{\text{cone}}\,(A+P)$. Then there exist $\lambda_i \geq 0$, $t_i \in [0,1]$, $u_i \in P$ such that

$$\lambda_i(t_i(y \pm z) + (1-t_i)(y+x)) + u_i \to y. \tag{15}$$

Then

$$\begin{aligned}
\langle x^*, \lambda_i(t_i(y \pm z) + (1-t_i)(y+x)) + u_i\rangle \to \langle x^*, y\rangle = 0 \Rightarrow \\
\lambda_i(1-t_i)\,\langle x^*, x\rangle + \langle x^*, u_i\rangle \to 0 \Rightarrow \\
\lambda_i(1-t_i) \to 0 \;\text{ and }\; \langle x^*, u_i\rangle \to 0.
\end{aligned}$$

If there is a subsequence of $\{\lambda_i\}$ converging to 0 then we get from (15) that $u_i \to y$ (since λ_i is multiplied with a bounded vector). This implies that $y \in \overline{P} = P$ which contradicts $\langle x^*, y\rangle = 0$.

If there is a subsequence of $\{\lambda_i\}$ converging to a number $\lambda \in\ ]0, +\infty[$ then $t_i \to 1$ and we get from (15) that $u_i \to y - \lambda(y \pm z)$. Since P is closed, this implies that $y - \lambda(y \pm z) \in P$. But $\langle x^*, y - \lambda(y \pm z)\rangle = 0$ while $\langle x^*, u\rangle > 0$ for all $u \in P \setminus \{0\}$. Hence $y - \lambda(y \pm z) = 0$. This is impossible, in view of the linear independence of y and z.

It follows that $\lambda_i \to +\infty$. Then $t_i \to 1$, and from $\lambda_i(1-t_i) \to 0$ and (15) we obtain $\lambda_i t_i(y \pm z) + u_i \to y$. Thus, $y \pm z + \frac{u_i}{\lambda_i t_i} \to 0$ and $\frac{u_i}{\lambda_i t_i} \to -(y \pm z)$. However, $\frac{u_i}{\lambda_i t_i} \in P$ thus its limit should be in P. As before, this should imply that $y \pm z = 0$ which again contradicts the linear independence of y and z.

Thus, $y \notin \overline{\text{cone}}\,(A+P)$. Since $y \in \text{co}\,(\overline{\text{cone}}\,(A+P))$, we deduce that $\overline{\text{cone}}\,(A+P)$ is not convex. This contradicts (a).

To show that (b) implies (a), we simply remark that if $\text{cone}\,(A) + P$ is convex then its closure $\overline{\text{cone}\,(\text{A}) + \text{P}}$ is convex, and this is equal to $\overline{\text{cone}}\,(A+P)$ by Proposition 2.1(d). The same proposition shows that (c) implies (a). Finally, (d) implies (b) and (c) by Theorem 4.1. $\square$

Remark 4.3 The assumption int $P^* \neq \emptyset$ (which corresponds to pointedness of P when Y is finite-dimensional) cannot be removed. Indeed, let $P = \{y \in Y : \langle p^*, y\rangle \geq 0\}$ where $p^* \in Y^* \backslash \{0\}$. Then $P^* = \operatorname{cone}(\{p^*\})$, int $P^* = \emptyset$. For any nonempty $A \subseteq Y$, the set $A + \operatorname{int} P$ is convex. Thus, (c) in Theorem 4.2 holds independently of the dimension of the space Y.

5 Some applications

5.1 Characterizing the zero (Lagrangian) duality gap

We now obtain various equivalent conditions to the zero (Lagrangian) duality gap for a class of nonconvex minimization problems under a Slater-type condition.

Let us consider the following constrained minimization problem

$$\mu \doteq \inf_{x \in K} f(x), \tag{16}$$

where $K \doteq \{x \in C : \; g(x) \in -P\}$, C is a nonempty subset of a real locally convex topological vector space X, $f : C \to \mathbb{R}$, and $g : C \to Y$, with Y as before and $P \subseteq Y$ is a closed convex cone with nonempty interior. Let us introduce the Lagrangian

$$L(\lambda^*, x) = f(x) + \langle \lambda^*, g(x)\rangle.$$

Obviously,

$$\mu \geq \inf_{x \in C} L(\lambda^*, x) \;\; \forall \, \lambda^* \in P^*. \tag{17}$$

We set

$$A \doteq \Big\{(f(x) - \mu, g(x)) \in \mathbb{R} \times Y : \; x \in C\Big\}.$$

Theorem 5.1 *Let us consider problem (16). If μ is finite and the Slater-type condition that for some $x_0 \in C$, $\langle y^*, g(x_0)\rangle < 0$ for all $y^* \in P^* \setminus \{0\}$ holds, then the following assertions are equivalent:*

(a) *there exists a Lagrange multiplier $\lambda^* \in P^*$ such that*

$$\inf_{x \in K} f(x) = \inf_{x \in C} L(\lambda^*, x);$$

(b)

$$\inf_{x \in K} f(x) = \max_{\lambda^* \in P^*} \inf_{x \in C} L(\lambda^*, x);$$

(c) $\operatorname{cone}(A + \operatorname{int}(\mathbb{R}_+ \times P))$ *is pointed.*

Proof. $(a) \Longleftrightarrow (b)$: One implication is obvious. From (a) it follows that

$$\mu \leq \max_{\lambda^* \in P^*} \inf_{x \in C} L(\lambda^*, x),$$

which together with (17) imply (b).
$(c) \Longrightarrow (a)$: Applying Theorem 3.2 we infer that $\mathrm{co}(A) \cap (-\mathrm{int}(\,\mathbb{R}_+ \times P)) = \emptyset$. By the convex separation theorem, we obtain $\gamma^* \geq 0$ and $\lambda^* \in P^*$, not both zero, satisfying

$$\gamma^* f(x) + \langle \lambda^*, g(x) \rangle \geq \gamma^* \mu \quad \forall\, x \in C. \tag{18}$$

If $\gamma^* = 0$, then $0 \neq \lambda^* \in P^*$ and $\langle \lambda^*, g(x) \rangle \geq 0$ for all $x \in C$, contradicting the Slater-type condition. Therefore, we may assume $\gamma^* = 1$ in (18). Hence,

$$f(x) + \langle \lambda^*, g(x) \rangle \geq \mu \quad \forall\, x \in C, \tag{19}$$

which implies

$$\inf_{x \in C} L(\lambda^*, x) \geq \mu.$$

This together with (17) yield the desired result.
$(a) \Longrightarrow (c)$: From (a), (19) holds, and this amounts to writing

$$\langle (1, \lambda^*), (f(x) - \mu, g(x)) \rangle \geq 0 \quad \forall\, x \in C.$$

We then apply Theorem 3.2 to get (c). □

5.2 Characterizing weakly efficient solutions through scalarization

Let X be a real vector space $K \subseteq X$ a convex set and Y a real locally convex topological vector space. Given a vector mapping $F : K \to Y$, we consider the problem of finding

$$\bar{x} \in K : \; F(x) - F(\bar{x}) \notin -\mathrm{int}\ P, \quad \forall\, x \in K,$$

where $P \subseteq Y$ is a closed convex cone such that $\mathrm{int}\ P \neq \emptyset$ (see Section 3). The set of such $\bar{x}$ is denoted by E_w, and its elements are termed weakly efficient solutions. Clearly

$$\bar{x} \in E_w \Longleftrightarrow (F(K) - F(\bar{x})) \cap (-\mathrm{int}\ P) = \emptyset.$$

For a real-valued function h, by $\mathrm{argmin}_K h$ we mean the set of minimum points of h on K.

The next theorem is a direct consequence of Corollary 3.3 with $G(x) = F(x) - F(\bar{x})$.

Theorem 5.2 *Let $K \subseteq X$ be a convex set and F, P as above. The following assertions are equivalent:*

(*a*)
$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq 0} \operatorname{argmin}_K \langle p^*, F(\cdot) \rangle;$$

(*b*) $\operatorname{cone}(F(K) - F(\bar{x}) + \operatorname{int} P)$ *is pointed.*

In case $Y = \mathbb{R}^2$, we get the following theorem whose proof follows from Theorem 4.1.

Theorem 5.3 *Let $K \subseteq X$ be a convex set and F, P as above with $Y = \mathbb{R}^2$. Then the following assertions are equivalent:*

(*a*)
$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq 0} \operatorname{argmin}_K \langle p^*, F(\cdot) \rangle;$$

(*b*) $\bar{x} \in E_w$ *and* $\operatorname{cone}(F(K) - F(\bar{x}) + \operatorname{int} P)$ *is convex.*

Notice that the cone appearing in (*b*) of the preceding theorem may be substituted by others cones by virtue of Theorem 4.1.

Remark 5.4 Some sufficient and in some situations also necessary conditions to get $E_w \neq \emptyset$ are established in [7, 8].

5.3 Characterizing the Fritz-John type optimality conditions in vector optimization

For simplicity we now consider X to be a real normed vector space. It is well known that if $\bar{x}$ is a local minimum point (in the usual sense) for the real-valued differentiable function F on K, then
$$\nabla F(\bar{x}) \in (T(K; \bar{x}))^*. \tag{20}$$
Here, $T(C; \bar{x})$ denotes the contingent cone of C at $\bar{x} \in C$, defined as the set of vectors v such that there exist $t_k \downarrow 0$, $v_k \in X$, $v_k \to v$ such that $\bar{x} + t_k v_k \in C$ for all k; C^* denotes the (positive) polar cone of C.

It is now our purpose to extend the previous optimality condition to the vector case without smoothness assumptions. More precisely, let $K \subseteq X$ be closed and consider a mapping $F : K \to \mathbb{R}^n$. Given a closed convex cone $P \subseteq \mathbb{R}^n$ with nonempty interior, a vector $\bar{x} \in K$ is a local weakly efficient solution for F on K, if there exists an open neighborhood V of $\bar{x}$ such that

$$(F(K \cap V) - F(\bar{x})) \cap (-\text{int } P) = \emptyset. \tag{21}$$

Following [15], we say that a function $h : X \to \mathbb{R}$ admits a Hadamard directional derivative at $\bar{x} \in X$ in the direction v if

$$\lim_{(t,u)\to(0^+,v)} \frac{h(\bar{x}+tu) - h(\bar{x})}{t} \in \mathbb{R}.$$

In this case, we denote such a limit by $dh(\bar{x}; v)$.

If $F = (f_1, \ldots, f_n)$, we set

$$\mathcal{F}(v) \doteq ((df_1(\bar{x}; v), \ldots, df_n(\bar{x}; v)), \quad \mathcal{F}(T(K; \bar{x})) = \{\mathcal{F}(v) \in \mathbb{R}^n : \ v \in T(K; \bar{x})\}.$$

It is known that if $df_i(\bar{x}; \cdot)$, $i = 1, \ldots, n$ do exist in $T(K; \bar{x})$, and $\bar{x} \in K$ is a local weakly efficient solution for F on K, i.e., $\bar{x}$ satisfies (21), then (see for instance Lemma 3.2 of [15])

$$(df_1(\bar{x}; v), \ldots, df_n(\bar{x}; v)) \in \mathbb{R}^n \setminus -\text{int } P, \quad \forall\, v \in T(K; \bar{x}),$$

or equivalently, $\mathcal{F}(T(K; \bar{x})) \cap (-\text{int } P) = \emptyset$. The following theorems provide complete characterizations for the validity of (a) as a necessary condition for $\bar{x}$ to be a local weakly efficient solution for F on K.

Theorem 5.5 *Let $K \subseteq X$ be a closed set, $P \subseteq \mathbb{R}^n$ be a closed convex cone such that $\text{int } P \neq \emptyset$, and $F : K \to \mathbb{R}^n$ be a mapping. Set $F = (f_1, \ldots, f_n)$ and assume that $\bar{x} \in K$ and $df_i(\bar{x}; \cdot)$, $i = 1, \ldots, n$ do exist in $T(K; \bar{x})$. Then, the following assertions are equivalent:*

- (a) $\exists\, (\alpha_1^*, \ldots, \alpha_n^*) \in P^* \setminus \{0\}, \quad \alpha_1^* df_1(\bar{x}, v) + \ldots + \alpha_n^* df_n(\bar{x}, v) \geq 0 \ \ \forall\, v \in T(K; \bar{x});$
- (b) $\text{cone}(\mathcal{F}(T(K; \bar{x})) + \text{int } P)$ *is pointed.*

Proof. We apply Corollary 3.3 to obtain the desired result. □

When $Y = \mathbb{R}^2$, more precise formulations can be obtained from Theorem 4.1.

Theorem 5.6 *Let $K \subseteq X$ be a closed set, $P \subseteq \mathbb{R}^2$ be a closed convex cone such that $\text{int } P \neq \emptyset$. Set $F = (f_1, f_2)$ and assume that $\bar{x} \in K$ and $df_i(\bar{x}; \cdot)$, $i = 1, 2$ do exist in $T(K; \bar{x})$. Then, the following assertions are equivalent:*

- (a) $\exists\, (\alpha_1^*, \alpha_2^*) \in P^* \setminus \{0\}, \quad \alpha_1^* df_1(\bar{x}, v) + \alpha_2^* df_2(\bar{x}, v) \geq 0 \ \ \forall\, v \in T(K; \bar{x});$

(*b*) $\mathcal{F}(T(K;\bar{x})) \cap (-\text{int } P) = \emptyset$ *and* $\text{cone}(\mathcal{F}(T(K;\bar{x})) + \text{int } P)$ *is convex.*

Remark 5.7 When $P = \mathbb{R}^n_+$ and $f_1, \ldots, f_n$ are differentiable, Part (*a*) in Theorem 5.5 can be written as

$$\text{co}(\{\nabla f_i(\bar{x}) : \ i = 1, \ldots, n\}) \cap (T(K;\bar{x}))^* \neq \emptyset, \tag{22}$$

which is the natural extension of (20). However, we have to point to out that (22) is not in general a necessary optimality condition for $\bar{x}$ to be a local weakly efficient solution. This is shown in $\mathbb{R}^2$ by the example taken from [2], see also [4, 18] for additional discussion:

$$K = \{(x_1, x_2) : \ (x_1 + 2x_2)(2x_1 + x_2) \leq 0\}, \ f_i(x_1, x_2) = x_i, \ \bar{x} = (0, 0).$$

In this case $T(K;\bar{x}) = K$, which is nonconvex, thus $(T(K;\bar{x}))^* = \{(0,0)\}$, and therefore (22) does not hold since $\text{co}\left(\{\nabla f_1(\bar{x}), \nabla f_2(\bar{x})\}\right) = \text{co}\left\{(1,0),(0,1)\right\}$. Notice also that

$$\text{cone}(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t \geq 0} t(T(K;\bar{x}) + \mathbb{R}^2_+)$$

is nonconvex. On the other hand, due to the linearity of $\mathcal{F}$ (when f_1 and f_2 are differentiable), if $T(K;\bar{x})$ is convex then

$$\text{cone}(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t \geq 0} t(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^2_+)$$

is also convex. This fact was point out earlier in [17] (see also [4]). Therefore, (22) holds if $T(K;\bar{x})$ is convex. The following example shows that the necessary optimality condition (22) may be true without the convexity of $T(K;\bar{x})$. Take the same mapping F as before and

$$K = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 x_2 = 0\}, \ \bar{x} = (0, 0).$$

Then, (22) holds since in this case, $T(K;\bar{x}) = K$, $(T(K;\bar{x}))^* = \mathbb{R}^2_+$ and

$$\text{co}\left(\{\nabla f_1(\bar{x}), \nabla f_2(\bar{x})\}\right) = \text{co}\left\{(1,0),(0,1)\right\}.$$

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References

[1] Adán M. and Novo V. (2002), Optimality conditions for vector optimization problems with generalized convexity in real linear spaces, *Optimization*, 51, 73–91.

[2] Aghezzaf B. and Hachimi M. (2001), On a gap between multiobjective optimization and scalar optimization, *J. of Optimization Theory and Applications*, 109, 431–435.

[3] Borwein J. and Goebel R. (2003), Notions of relative interior in Banach spaces, *J. of Mathematical Sciences*, 115, 2542-2553.

[4] Castellani M. and Pappalardo M. (2001), About a gap between multiobjective optimization and scalar optimization, *J. of Optimization Theory and Applications*, 109, 437–439.

[5] Deng S. (2003), Coercivity properties and well posedness in vector optimization, *RAIRO Operations Research*, 37, 195–208.

[6] Fliege J. (2004), Gap-free computation of Pareto-points by quadratic scalarizations, *Mathematical Methods of Operations Research*, 59, 69–89.

[7] Flores-Bazán F. (2002), Ideal, weakly efficient solutions for vector optimization problems, *Mathematical Programming, Ser. A.*, 93, 453–475.

[8] Flores-Bazán F. and Vera C. (2006), Characterization of the nonemptiness and compactness of solution sets in convex/nonconvex vector optimization, *J. of Optimization Theory and Applications*, 130. In Press.

[9] Hu S. and Papageorgiou N.S. (1997), *Handbook of Multivalued Analysis (Volume I)*, Kluwer Academic Publishers.

[10] Jeyakumar V., Oettli W. and Natividad M. (1993), A solvability theorem for a class of quasiconvex mappings with applications to optimization, *J. of Mathematical Analysis and Applications*, 179, 537–546.

[11] Peressini A.L. (1967), *Ordered Topological Vector Spaces*, Harper and Row, New York and London.

[12] Rockafellar R.T. and Wets R.J-B. (1998), *Variational Analysis*, Springer-Verlag, Berlin Hedelberg.

[13] Sach P.H. (2003), Nearly subconvexlike set-valued maps and vector optimization problems, *J. of Optimization Theory and Applications*, 119, 335–356.

[14] Song W. (1998), Duality in Set-Valued Optimization, *Dissertationes Mathematicae*, 375, 1–67.

[15] Taa A. (1999), Optimality conditions for vector mathematical programming via a theorem of the alternative, *J. of Mathematical Analysis and Applications*, 233, 233–245.

[16] Tanaka T. (1994), General quasiconvexities, cones saddle points and minimax theorem for vector-valued functions, *J. of Optimization Theory and Applications*, 81, 355–377.

[17] Wang S.Y. (1988), A note on optimality conditions in multiobjective programming, *Systems Science and Mathematical Sciences*, 1, 184–190.

[18] Wang S.Y. and Yang F.M. (1991), A gap between multiobjective optimization and scalar optimization, *J. of Optimization Theory and Applications*, 68, 389–391.

[19] Yang X.M., Li D. and Wang S.Y. (2001), Nearly-subconvexlikeness in vector optimization with set-valued functions, *J. of Optimization Theory and Applications*, 110, 413–427.

[20] Yang X.M., Yang X.Q and Chen G.Y. (2000), Theorems of the alternative and optimization with set-valued maps, *J. of Optimization Theory and Applications*, 107, 627–640.

[21] Zeng R. (2002), A general Gordan alternative theorem with weakened convexity and its application, *Optimization*, 51, 709–717.