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# First and Second Order Asymptotic Analysis with Applications in Quasiconvex Optimization

**F. Flores-Bazán · N. Hadjisavvas · F. Lara · I. Montenegro**

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F. Flores-Bazán, Corresponding author

Departamento de Ingeniería Matemática, Universidad de Concepción, Concepción, Chile

fflores@ing-mat.udec.cl

N. Hadjisavvas

Mathematics and Statistics Department, King Fahd University of Petroleum and Minerals, Dhahran, Kingdom of Saudi Arabia

nhadjisavvas@kfupm.edu.sa

F. Lara

Departamento de Matemáticas, Universidad de Tarapacá, Arica, Chile

felipelaraobreque@gmail.com

I. Montenegro

Departamento de Ingeniería Matemática, Universidad de Concepción, Concepción, Chile

ivanmontenegro@udec.cl

**Abstract** We use asymptotic analysis to describe in a more systematic way the behavior at the infinity of functions in the convex and quasiconvex case. Starting from the formulae for the first and second order asymptotic function in the convex case, we introduce similar notions suitable for dealing with quasiconvex functions. Afterwards, by using such notions, a class of quasiconvex vector mappings under which the image of a closed convex set is closed, is introduced; we characterize the nonemptiness and boundedness of the set of minimizers of any lsc quasiconvex function; finally, we also characterize boundedness from below, along lines, of any proper and lsc function.

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## 1 Introduction

Asymptotic analysis involves a description of the behaviour of a mathematical object at infinity. Usually it concerns a set, or a function via its epigraph. When a minimization problem is considered, convexity is the desired condition since any local property has a global character: for example, any local minimizer is global, and first order necessary optimality conditions become also sufficient. Under lack of convexity an analysis of the behaviour of unbounded minimizing sequences is necessary, and then once we normalize them, their limit directions need to be compared with those stemming from the epigraph of the objective function.

In general, the existence issue in nonconvex minimization problems requires a global knowledge of the objects. However, quasiconvex objective functions still provide a good instance where we may apply the same tools, suitably modified, coming from convex situations. This paper goes in that direction.

The paper is organized as follows. Section 2 collects some basic definitions, some of them well known like (first order) asymptotic cones and functions; their second order counterparts are also recalled, in the general case. Then, first and second order asymptotic cones and functions, which seem to be suitable for dealing with quasiconvex functions, are introduced.

Section 3 shows some applications of the notions introduced in Section 2: we identify a new class of quasiconvex vector mappings and provide a sufficient condition under which the image, via a mapping belonging to that class, of a closed convex set is closed; when minimizing a quasiconvex function, we characterize the nonemptiness and boundedness of the optimal solution set; finally, we also characterize boundedness from below, along lines, of any lower semicontinuous function.

## 2 Some Preliminaries and Basic Definitions

We denote the duality pairing between two elements of $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. Given $u \in \mathbb{R}^n$, the subspace generated by the vector $u$ is denoted by $\mathbb{R}u$. Let $K \subseteq \mathbb{R}^n$, its affine hull, denoted by aff $K$, is the smallest affine set containing $K$; its boundary by bd $K$; its topological interior by int $K$. In addition, its relative interior, denoted by ri $K$, is the interior with respect to its affine set.

## 2.1 First and Second Order Asymptotic Cones and Functions: The General Case

For $K \subseteq \mathbb{R}^n$, its first order asymptotic cone (or just asymptotic cone) is defined by

$$K^\infty := \{u \in \mathbb{R}^n : \ \exists\, t_k \to +\infty, \ \exists\, x_k \in K, \ \frac{x_k}{t_k} \to u\}.$$

In case $K \subseteq \mathbb{R}^n$ is a closed and convex set, it is known that the concept of asymptotic cone or recession cone (see [1,2]) is equivalent to

$$K^\infty = \{u \in \mathbb{R}^n : \ x_0 + \lambda u \in K, \ \forall\, \lambda \geq 0\}, \text{ for any } x_0 \in K. \tag{1}$$

The lineality space of $K$ is given by lin $K := K^\infty \cap (-K^\infty)$.

Given any function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the epigraph of $f$ is defined by epi $f := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \ f(x) \leq t\}$, its effective domain is given by dom $f := \{x \in \mathbb{R}^n : \ f(x) < +\infty\}$, and for a given $\lambda \in \mathbb{R}$, we denote by $S_\lambda(f) := \{x \in \mathbb{R}^n : \ f(x) \leq \lambda\}$ the sublevel set of $f$ at the height $\lambda$. The first order asymptotic function (or just asymptotic function) of $f$ is defined as the function $f^\infty : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ satisfying

$$\text{epi } f^\infty := (\text{epi } f)^\infty.$$

From this, clearly

$$f^\infty(u) = \inf \left\{ \liminf_{n \to +\infty} t_n f(\frac{x_n}{t_n}) : \ t_n \downarrow 0, x_n \to u \right\}.$$

When $f$ is a convex and lower semicontinuous function, we have for all $x_0 \in f^{-1}(\mathbb{R})$,

$$f^\infty(u) = \lim_{t \to +\infty} \frac{f(x_0 + tu) - f(x_0)}{t} = \sup_{t > 0} \frac{f(x_0 + tu) - f(x_0)}{t}. \tag{2}$$

Note that, if $f$ is only convex but not necessarily lsc, then (2) also holds, if we take $x_0 \in \operatorname{ri}\operatorname{dom} f$.

A notion of "second order asymptotic cone" was introduced in [3] for generalized sets. Characterizations for convex sets were given in [4].

**Definition 2.1** Given a nonempty set $K \subseteq \mathbb{R}^n$ and $u \in \mathbb{R}^n$, we say that $v \in \mathbb{R}^n$ is a second order asymptotic direction of $K$ at $u$ if there are sequences $x_k \in K$, $s_k$ and $t_k \in \mathbb{R}$, with $s_k, t_k \to +\infty$ such that,

$$v := \lim_{k\to+\infty} \left( \frac{x_k}{s_k} - t_k u \right). \tag{3}$$

The set of all such elements $v$ is denoted by $K^{\infty 2}[u]$.

The set $K^{\infty 2}[u]$ is a cone, termed the second order asymptotic cone of $K$ at $u$. It is nonempty exactly when $u \in K^{\infty}$. If $u = 0$ then $K^{\infty 2}[0] = K^{\infty}$. Since $K^{\infty 2}[u]$ can be expressed by the outer limit

$$K^{\infty 2}[u] = \limsup_{s,t\uparrow+\infty} \left( \frac{K}{s} - tu \right), \tag{4}$$

it is closed.

We recall some basic properties of the second order asymptotic cone presented in [3] and developed in [4].

**Proposition 2.1** *Let $K \subseteq \mathbb{R}^n$, then the following assertions hold:*

(a) *If $K_0 \subseteq K$, then $(K_0)^{\infty 2}[u] \subseteq K^{\infty 2}[u]$ for all $u \in (K_0)^{\infty}$.*

(b) *$(K + z)^{\infty 2}[u] = K^{\infty 2}[u]$, for all $u \in K^{\infty}$ and $z \in \mathbb{R}^n$.*

(c) *$K^{\infty 2}[u] + \mathbb{R}u = K^{\infty 2}[u]$, for all $u \in K^{\infty}$.*

(d) *If $u \in \operatorname{ri} K^{\infty}$, then $\operatorname{aff} K^{\infty} = K^{\infty} - K^{\infty} \subseteq K^{\infty 2}[u]$.*

(*e*) *Let* $\{K_i\}_{i\in I}$ *be a family of sets and* $u \in \mathbb{R}^n$, *then*

$$\bigcup_{i\in I}(K_i)^{\infty 2}[u] \subseteq (\bigcup_{i\in I} K_i)^{\infty 2}[u].$$

*The equality holds when* $|I| < +\infty$.

(*f*) *Let* $\{K_i\}_{i\in I}$ *be a family of sets satisfying* $\mathrm{ri}\ \bigcap_{i\in I} K_i \neq \emptyset$ *and let* $u \in (\bigcap_{i\in I} K_i)^{\infty}$ *then*

$$(\bigcap_{i\in I} K_i)^{\infty 2}[u] \subseteq \bigcap_{i\in I}(K_i)^{\infty 2}[u].$$

*The equality holds when every* $K_i$ *is convex and* $|I| < +\infty$.

In the case when $K$ is a convex subset of $\mathbb{R}^n$, the authors in [4] give a characterization of $K^{\infty 2}[u]$ which reminds the one for $K^{\infty}$ given by (1).

**Proposition 2.2** *For a nonempty set* $K \subseteq \mathbb{R}^n$ *with* $x \in \mathrm{ri}\ K$, *consider the following assertions:*

(*a*) $u \in K^{\infty}$ *and* $v \in K^{\infty 2}[u]$;

(*b*) $\forall\ s > 0,\ \exists\ \bar{t} > 0$ *such that* $x + tu + sv \in K,\ \forall\ t > \bar{t}$;

(*c*) $\exists\ s_n, t_n \to +\infty$ *such that* $x + s_n t_n u + s_n v \in K$.

*Then we have*

$$b) \Rightarrow c) \Rightarrow a).$$

*In addition, if* $K$ *is convex, then the three assertions are equivalent.*

Note that, for a convex set $K$, if $(b)$ is true for some $x \in \mathrm{ri}\ K$, then this implies $(a)$, which in turn implies $(b)$ and $(c)$ for every $x \in \mathrm{ri}\ K$. Hence, if $(b)$ is true for some $x \in \mathrm{ri}\ K$, then it is true for all. Furthermore, the point

$x \in \text{ri } K$ cannot be replaced by $x \in K$ in the general case as [4, Example 3.9] shows.

It was proved in [4] that the first and second order asymptotic cones coincide in some cases, for example, if $u \in \text{lin } K$. We recall the result for the convenience of the reader.

**Proposition 2.3** *If $K \subseteq \mathbb{R}^n$ is a convex set, then $K^\infty \subseteq K^{\infty 2}[u]$ for all $u \in K^\infty$. The equality holds if and only if $u \in K^\infty \cap (-K^\infty)$.*

*Proof* Given $x \in \text{ri } K$, we note that for every $u, v \in K^\infty$, and for every $s, t > 0$, $x + su \in \text{ri } K$ so $x + tu + sv \in K$. Thus by the characterization of Proposition 2.2, $v \in K^{\infty 2}[u]$.

Let $x \in \text{ri } K$ and $v \in K^{\infty 2}[u]$. For every $s > 0$, we can find $t$ such that $x+tu+sv \in K$. Since $-u \in K^\infty$, $(x + tu + sv)+t(-u) \in K$. Thus $x+sv \in K$ for all $s > 0$, so $v \in K^\infty$. This shows that $K^{\infty 2}[u] \subseteq K^\infty$.

Finally, if $K^{\infty 2}[u] \subseteq K^\infty$, then from Proposition 2.1(c) we deduce that $u \in K^\infty \cap (-K^\infty)$. □

Note that the previous result is in general false for nonconvex sets, as the next example shows: take the nonconvex set $K = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ and $u = (1, 0)$. Then, $K = K^\infty$ and $K^{\infty 2}[u] = \mathbb{R} \times \{0\}$.

If the first and second order asymptotic cones are equal, the second order asymptotic cone does not provide any new information of the set.

A class of important closed and convex sets in connection to Proposition 2.3 is that of well-positioned. We recall that, a closed and convex set is well-positioned if $K^\infty$ is pointed, see [5]. Thus, in case $K$ is not well-positioned, there exists $u \in K^\infty \cap -K^\infty$ such that $K^\infty = K^{\infty 2}[u]$.

**Definition 2.2** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Let $u \in \mathbb{R}^n$ be such that $f^\infty(u) \in \mathbb{R}$. Then the second order asymptotic function of $f$ at $u$, denoted by $f^{\infty 2}(u;\cdot)$, is defined by

$$\text{epi } f^{\infty 2}(u;\cdot) := (\text{epi } f)^{\infty 2}[(u, f^\infty(u))]. \tag{5}$$

Since $K^{\infty 2}$ is a closed cone, then $f^{\infty 2}(u;\cdot)$ is lsc and positively homogeneous. Furthermore, it was proved in [4] that $f^{\infty 2}(u;0) = 0$ or $-\infty$, while $f^{\infty 2}(u;0) = 0$ if and only if $f^{\infty 2}(u;\cdot)$ is proper.

From (5) we derive the next formula, see [4] for details. Let $u \in \mathbb{R}^n$ be such that $f^\infty(u)$ is finite. Then for every $v \in \mathbb{R}^n$,

$$\begin{aligned} f^{\infty 2}(u;v) = \inf \Bigg\{ \liminf_{k\to\infty} \left( \frac{f(x_k)}{s_k} - t_k f^\infty(u) \right) : \\ x_k \in \text{dom } f, s_k, t_k \to +\infty,\ \frac{x_k}{s_k} - t_k u \to v \Bigg\}. \end{aligned} \tag{6}$$

Note that, in [3], the second order asymptotic function was defined directly through formula (6), was called "lower second order asymptotic function"and denoted by $R''_{-} f(u;\cdot)$.

When $f$ is a proper convex function and $x \in \text{ri}\,\text{dom } f$, then for every $u$ such that $f^\infty(u)$ is finite and $v \in (\text{dom } f)^{\infty 2}[u]$, we have

$$f^{\infty 2}(u;v) = \sup_{s>0} \inf_{t>0} \frac{f(x+tu+sv) - tf^\infty(u) - f(x)}{s}, \tag{7}$$

$$= \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{f(x+tu+sv) - tf^\infty(u) - f(x)}{s}. \tag{8}$$

For several comments, remarks and examples about convex sets and functions, see [4].

By Proposition 2.3, if $f$ is convex, proper and lsc function, and epi $f^\infty$ is not a pointed cone, then there exists $(u, f^\infty(u)) \in (\text{epi } f)^\infty$, such that

$$\text{epi } f^\infty = (\text{epi } f)^\infty = (\text{epi } f)^{\infty 2}[(u, f^\infty(u))] = \text{epi } f^{\infty 2}(u; \cdot),$$

thus $f^\infty(v) = f^{\infty 2}(u; v)$ for all $v \in K^{\infty 2}[u]$.

For every function for which epi $f^\infty$ is a pointed cone, the second order asymptotic function provides additional information of the behavior at the infinity. This is the case, for example, of well-positioned functions or the "semibounded" functions, defined and developed in [6] with several applications in economics.

## 2.2 Variants of First and Second Order Asymptotic Functions

The usual definition of (first order) asymptotic function does not provide adequate information when the original function is quasiconvex. For that reason, many authors search for alternatives in such a situation. Some attempts appear in [7,8], see also the very recent contribution in [9]. A study of quasiconvex functions and possible notions of subdifferential were discussed in [10].

We start this section by defining another (first order) asymptotic cone, very close to the usual concept with several applications to generalized differentiability, called the incident asymptotic cone; see [1,11,12].

**Definition 2.3** Let $K \subseteq \mathbb{R}^n$ be a nonempty set. The incident asymptotic cone of $K$ is the set

$$K^{i\infty} := \{u \in \mathbb{R}^n : \ \forall\ t_k \to +\infty,\ \exists\ x_k \in K,\ \frac{x_k}{t_k} \to u\}. \tag{9}$$

It is clear that $K^{i\infty}$ is a closed cone and $K^{i\infty} \subseteq K^{\infty}$. We say $K$ is *asymptotically regular* (*asymptotable* in [13]) if $K^{i\infty} = K^{\infty}$. By [1, Proposition 2.1.3] we know that every convex set is asymptotically regular. Moreover, if $K$ is a closed cone then $K^{i\infty} = K (= K^{\infty})$, and so it is asymptotically regular. Other sufficient condition for having $K^{i\infty} = K^{\infty}$ may be found in [13].

Observe that, there are unbounded sets for which $K^{i\infty} = \{0\}$. Indeed, simply take $K = \{2^{2m} \in \mathbb{R} : \ m = 0, 1, 2, \ldots\}$, for which we obtain, $K^{\infty} = \mathbb{R}_+$ and $K^{i\infty} = \{0\}$, see [14, Chapter 1, Section 2, p. 13].

The following functions, taken from [9] motivated by [15], describe the behaviour of $f$ along any direction by taking into account their values everywhere, and not only points at infinity. This is natural, because for nonconvex functions the nonemptyness and the structure of the set of minimizers does not depend only on the behaviour of the function at infinity.

**Definition 2.4** For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, given $\lambda \in \mathbb{R}$ such that $S_\lambda(f) \neq \emptyset$, we consider the following asymptotic functions:

$$f^{\infty}(u; \lambda) := \sup_{x \in S_\lambda(f)} \sup_{t>0} \frac{f(x + tu) - \lambda}{t}, \tag{10}$$

$$f_q^{\infty}(u) := \sup_{\substack{x \in \mathrm{dom}\ f \\ t>0}} \frac{f(x + tu) - f(x)}{t}. \tag{11}$$

In addition, the function $f^{i\infty} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, termed the incident asymptotic function, is defined as the function satisfying

$$\text{epi } f^{i\infty} := (\text{epi } f)^{i\infty}. \tag{12}$$

From the above definitions, we observe that $f(\cdot, \lambda)$ and $f_q^\infty$ are positively homogeneous functions.

In case when $\lambda = f(\overline{x})$ for some $\overline{x} \in \text{dom } f$, we simply write (as in [9])

$$f^\infty(u; \overline{x}) = f^\infty(u; f(\overline{x})) = \sup_{x \in S_{f(\overline{x})}(f)} \sup_{t>0} \frac{f(x+tu) - f(\overline{x})}{t}.$$

When $f$ is a quasiconvex function, it is easy to see that $f^{i\infty}$, $f_q^\infty$ and $f^\infty(\cdot; \lambda)$ are quasiconvex by [10, Proposition 11.1] and [9, Proposition 3.28], respectively (Notice that, the quasiconvexity of $f$ implies the quasiconvexity of the function

$$\frac{f(x+t\cdot) - f(x)}{t}).$$

Now, it remains to answer the question whether $f^\infty$ is quasiconvex if $f$ is so. Unfortunately, we were able to answer in the affirmative only in the one-dimensional case, and no example showing the negative in higher dimension was found. We first mention the following proposition, whose easy proof is omitted.

**Proposition 2.4** *A function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty, -\infty\}$ is quasiconvex if and only if there exists an interval $I$ of the form $]-\infty, b[$ or $]-\infty, b]$, where $b \in ]-\infty, +\infty]$, such that $f$ is nonincreasing on $I$ and nondecreasing on its complement.*

The desired result follows next.

**Proposition 2.5** *If $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a quasiconvex function, then $f^\infty : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is quasiconvex.*

*Proof* For a quasiconvex function $f$, one has either $f(x) \geq f(0)$ for all $x > 0$, or $f(x) \geq f(0)$ for all $x \leq 0$ (or both); indeed, if we assume that this is not the case, then we get some $x_1 > 0$ and some $x_2 < 0$ such that $f(x_1) < f(0)$ and $f(x_2) < f(0)$, contradicting quasiconvexity.

Hence we may assume, for instance, that $f(x) \geq f(0)$ for all $x > 0$. Then we get $f^\infty(1) \geq 0$, so $f^\infty$ is nondecreasing on $]0, \infty[$.

If $f^\infty(0) = 0$, then $f^\infty$ is nondecreasing on $[0, \infty[$. Since it is either decreasing or nondecreasing on $] - \infty, 0]$, we deduce by Proposition 2.4 that $f^\infty$ is quasiconvex.

Now assume that $f^\infty(0) = -\infty$. If $f^\infty(x) \geq 0$ for all $x < 0$ then by Proposition 2.4, $f^\infty$ is quasiconvex. So we may assume that $f^\infty(-1) < 0$. In this case, we will show that $f^\infty(-1) = -\infty$, so $f^\infty$ is nondecreasing on $\mathbb{R}$ and thus it is quasiconvex.

Since $f^\infty(0) = -\infty$, there exists a sequence $\{x_k\} \subseteq \operatorname{dom} f$ and $t_k \to +\infty$ such that $\frac{x_k}{t_k} \to 0$ and $\frac{f(x_k)}{t_k} \to -\infty$. It follows that $f(x_k) \to -\infty$ so we may assume that $f(x_k) < f(0)$ and $x_k < 0$.

Since $f^\infty(-1) < 0$, there exist sequences $\{y_k\} \subseteq \operatorname{dom} f$ and $s_k \to +\infty$ such that $\frac{y_k}{s_k} \to -1$ and $\lim_{k \to +\infty} \frac{f(y_k)}{s_k} = f^\infty(-1) < 0$. This implies that $y_k \to -\infty$ and $f(y_k) \to -\infty$. Set $z_k = x_k - t_k$. For each $k$ we may choose $k'$ such that $y_{k'} < z_k < x_k$ and $f(y_{k'}) < f(x_k)$. By quasiconvexity, $f(z_k) \leq f(x_k)$ for all $k$.

Then $\frac{z_k}{t_k} \to -1$ and $\frac{f(z_k)}{t_k} \leq \frac{f(x_k)}{t_k} \to -\infty$. Hence $f^\infty(-1) = -\infty$ as was to be proved. Then $f^\infty$ is quasiconvex. $\square$

Furthermore, if $f$ is proper, convex and lsc, then all these notions coincide with the usual asymptotic function $f^\infty$, that is, for all $u \in \mathbb{R}^n$ and for all $x \in \text{dom } f$, we have

$$f^\infty(u) = f_q^\infty(u) = f^\infty(u; x) = f^{i\infty}(u).$$

We will now establish another formula for $f_q^\infty$ in the general case under lower semicontinuity. To that purpose, some notions are needed. Recall that, for any function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the upper and lower Dini directional derivatives of $f$ at $x \in \text{dom } f$ in the direction $u \in \mathbb{R}^n$ are defined by

$$\begin{aligned} f^D(x; u) &:= \limsup_{t \to 0^+} \frac{f(x + tu) - f(x)}{t}, \\ f_D(x; u) &:= \liminf_{t \to 0^+} \frac{f(x + tu) - f(x)}{t}. \end{aligned}$$

If $f$ is lsc, then we know from the Diewert mean value theorem (see for example [16, Theorem 10.1]): for any $a, b \in \text{dom } f$, there exists $z \in [a, b[$ such that

$$f_D(z; b - a) \geq f(b) - f(a).$$

The new formula for $f_q^\infty$ in terms of upper or lower Dini directional derivatives is expressed next, which also provides another formula for $f^\infty$ when $f$ is convex.

**Proposition 2.6** *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be lsc with* dom $f$ *being nonempty and convex. Then*

$$f_q^\infty(u) = \sup_{x \in \mathrm{dom}\ f} f^D(x;u) = \sup_{x \in \mathrm{dom}\ f} f_D(x;u). \tag{13}$$

*Proof* Set $\alpha := \sup_{x \in \mathrm{dom}\ f} f^D(x;u)$ and $\beta := \sup_{x \in \mathrm{dom}\ f} f_D(x;u)$. Note that, for every $y \in \mathrm{dom}\ f$,

$$f^D(y;u) \le \sup_{t>0} \frac{f(y+tu) - f(y)}{t}.$$

Hence,

$$f_D(y;u) \le f^D(y;u) \le \sup_{x \in \mathrm{dom}\ f} \sup_{t>0} \frac{f(x+tu) - f(x)}{t} = f_q^\infty(u),\ \forall\, y \in \mathrm{dom}\ f,$$

from which follows $\beta \le \alpha \le f_q^\infty(u)$.

To show the reverse inequality, take $u \in \mathbb{R}^n$. Assume first that, for every $y \in \mathrm{dom}\ f$ and $t > 0$, $y + tu \in \mathrm{dom}\ f$. By Diewert's mean value theorem there exists $z \in [y, y+tu[$ such that

$$f(y+tu) - f(y) \le f_D(z;tu).$$

Since

$$f_D(z;tu) = t f_D(z;u) \le t \sup_{x \in \mathrm{dom}\ f} f_D(x;u),$$

it follows that

$$\frac{f(y+tu) - f(y)}{t} \le \beta,\ \forall\, y \in \mathrm{dom}\ f,\ t > 0.$$

Hence $f_q^\infty(u) \le \beta$, and the desired equalities in (13) are proved.

Now assume that for some $y \in \mathrm{dom}\ f$ and $t > 0$, $y + tu \notin \mathrm{dom}\ f$. Then $f_q^\infty(u) = +\infty$. Let $t_0 = \sup\{t : y + tu \in \mathrm{dom}\ f\} \in \mathbb{R}$. If $y + t_0 u \in \mathrm{dom}\ f$, then clearly $f_D(y + t_0u; u) = +\infty$ and $\beta = +\infty = \alpha$. If not, then $f(y + t_0u) = +\infty$ and by lower semicontinuity, $\lim_{t\to t_0^-} f(y + tu) = +\infty$. Using again Diewert's mean value theorem, it is easy to see that

$$\sup_{z\in[y,y+t_0u[} f_D(z; u) = +\infty,$$

so we find again $\beta = +\infty = \alpha$. Thus, in all cases, we have the equalities in (13). $\square$

For example, if $f$ is the increasing, thus quasiconvex function given by $f(x) = x + \sin x$, $x \in \mathbb{R}$, we get $f_q^\infty(1) = \sup_{x\in\mathbb{R}} f'(x) = 2$.

*Remark 2.1* (*i*) Note that, in spite of the fact that for a given $x \in \operatorname{dom} f$, in general

$$f^D(x; u) = \limsup_{t\to 0^+} \frac{f(x + tu) - f(x)}{t} \neq \sup_{t>0} \frac{f(x + tu) - f(x)}{t},$$

we still have

$$\sup_{x\in\operatorname{dom}\ f} f^D(x; u) = \sup_{x\in\operatorname{dom}\ f}\ \sup_{t>0} \frac{f(x + tu) - f(x)}{t}.$$

In fact, the proof shows that for every $x_0 \in \operatorname{dom} f$, if $l_{x_0} = \{x_0 + tu; t \geq 0\}$ then

$$\sup_{x\in l_{x_0}} f^D(x; u) = \sup_{x\in l_{x_0}}\ \sup_{t>0} \frac{f(x + tu) - f(x)}{t}.$$

(*ii*) When $f$ is convex and lsc, then $f_q^\infty = f^\infty$ so we have still another formula for $f^\infty$, that is, $f^\infty(u) = \sup_{x\in\operatorname{dom}\ f} f^D(x; u) = \sup_{x\in\operatorname{dom}\ f} f_D(x; u)$.

Following standard arguments, see [1, Proposition 2.5.1] or [15], we can prove the basic properties of $f^{i\infty}$ listed in the next proposition.

**Proposition 2.7** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function, then $f^{i\infty}$ is lsc and positively homogeneous; $f^{i\infty}(0) = 0$ or $-\infty$, and if $f^{i\infty}(0) = 0$ then $f^{i\infty}$ is proper.*

The incident asymptotic function may be computed by the following new formula. According to our best knowledge, no formula appears in the literature.

**Proposition 2.8** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then for all $u \in \mathbb{R}^n$*

$$f^{i\infty}(u) = \sup_{t_k \to +\infty} \inf_{\frac{x_k}{t_k} \to u} \limsup_{k \to +\infty} \frac{f(x_k)}{t_k}. \tag{14}$$

*Proof* ($\geq$) Denote by $h(u)$ the right-hand side of (14). Let $(u, \alpha) \in \text{epi } f^{i\infty}$, then for all $t_k \to +\infty$ there exists $(x_k, \alpha_k) \in \text{epi } f$ such that $\frac{(x_k, \alpha_k)}{t_k} \to (u, \alpha)$. Since $\frac{f(x_k)}{t_k} \leq \frac{\alpha_k}{t_k}$, then

$$\limsup_{k \to +\infty} \frac{f(x_k)}{t_k} \leq \limsup_{k \to +\infty} \frac{\alpha_k}{t_k} = \alpha,$$

thus,

$$\inf_{\frac{x_k}{t_k} \to u} \limsup_{k \to +\infty} \frac{f(x_k)}{t_k} \leq \alpha, \ \forall \ t_k \to +\infty,$$

then

$$\sup_{t_k \to +\infty} \inf_{\frac{x_k}{t_k} \to u} \limsup_{k \to +\infty} \frac{f(x_k)}{t_k} \leq \alpha,$$

thus $h(u) \leq \alpha$ for all $(u, \alpha) \in \text{epi } f^{i\infty}$, so $h(u) \leq f^{i\infty}(u)$.

($\leq$) If $h(u) = +\infty$, then the inequality is obvious. If $h(u) \in \mathbb{R}$, let $\varepsilon > 0$, then $h(u) < h(u) + \varepsilon$, thus for all $t_k \to +\infty$

$$\inf_{\frac{x_k}{t_k} \to u} \limsup_{k \to +\infty} \frac{f(x_k)}{t_k} < h(u) + \varepsilon.$$

Then for all $t_k \to +\infty$, there exists $\frac{x_k}{t_k} \to u$ such that

$$\limsup_{k\to+\infty} \frac{f(x_k)}{t_k} < h(u) + \varepsilon.$$

Let us consider,

$$\alpha_k := \begin{cases} t_k(h(u)+\varepsilon), & \text{if } \frac{f(x_k)}{t_k} \leq h(u)+\varepsilon, \\ f(x_k), & \text{if } \frac{f(x_k)}{t_k} > h(u)+\varepsilon. \end{cases}$$

Then, in any case, we have that $f(x_k) \leq \alpha_k$, so $(x_k, \alpha_k) \in \text{epi } f$. Since

$$\frac{(x_k, \alpha_k)}{t_k} \to (u, h(u)+\varepsilon),$$

then $(u, h(u)+\varepsilon) \in \text{epi } f^{i\infty}$, thus $f^{i\infty}(u) \leq h(u)+\varepsilon$ for all $\varepsilon > 0$. Consequently $f^{i\infty}(u) \leq h(u)$. $\square$

In addition, we also obtain the following relationship between $f^{\infty}_q$ and $f^{i\infty}$.

**Proposition 2.9** *For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, one obtains for all $u \in \mathbb{R}^n$*

$(a)$ $$f^{i\infty}(u) \leq \inf_{x\in\text{dom } f} \limsup_{t\to+\infty} \frac{f(x+tu)}{t}$$
$$\leq \inf_{x\in\text{dom } f} \sup_{t>0} \frac{f(x+tu)-f(x)}{t} \leq f^{\infty}_q(u); \tag{15}$$

$(b)$ *given $\lambda \in \mathbb{R}$ such that $S_\lambda(f) \neq \emptyset$,*

$$f^{\infty}(u;\lambda) \leq \sup_{x\in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu)-f(x)}{t} \leq f^{\infty}_q(u);$$

$(c)$ $f^{\infty}(u) \leq f^{\infty}_q(u)$;

$(d)$ $f^{\infty}(u) \leq f^{i\infty}(u)$.

*Proof* Let $x \in \operatorname{dom} f$ be given. For every $t_k \to +\infty$, set $y_k = x + t_k u$. Then $\frac{y_k}{t_k} \to u$. One obviously has

$$\limsup_{k \to +\infty} \frac{f(y_k)}{t_k} \leq \limsup_{t \to +\infty} \frac{f(x+tu)}{t},$$

hence for this sequence $\{t_k\}$,

$$\inf_{\frac{x_k}{t_k} \to u} \left\{ \limsup_{k \to +\infty} \frac{f(x_k)}{t_k} \right\} \leq \limsup_{t \to +\infty} \frac{f(x+tu)}{t}.$$

This is true for all $t_k \to +\infty$, so $f^{i\infty}(u) \leq \limsup_{t \to +\infty} \dfrac{f(x+tu)}{t}$. Since this is true for all $x$, we deduce the first inequality in (15). As for the second, we remark that for every $x \in \operatorname{dom} f$,

$$\limsup_{t \to +\infty} \frac{f(x+tu)}{t} = \limsup_{t \to +\infty} \frac{f(x+tu) - f(x)}{t} \leq \sup_{t>0} \frac{f(x+tu) - f(x)}{t}.$$

Thus,

$$\inf_{x \in \operatorname{dom} f} \limsup_{t \to +\infty} \frac{f(x+tu)}{t} \leq \inf_{x \in \operatorname{dom} f} \sup_{t>0} \frac{f(x+tu) - f(x)}{t} \leq f_q^{\infty}(u).$$

Assertions $(b)$, $(c)$ and $(d)$ are obvious. □

The following two examples show that some of the inequalities in the previous proposition may be strict.

*Example 2.1* ([9, Example 5.6]) Consider the quasiconvex and lsc function $f(x) = \frac{x}{1+x}$ if $x \geq 0$, and $f(x) = +\infty$ if $x < 0$. We obtain

$$f^{\infty}(u) = f^{i\infty}(u) = \begin{cases} 0, & \text{if } u \geq 0, \\ +\infty, & \text{if } u < 0. \end{cases}$$

$$f^{\infty}(u; \overline{x}) = \begin{cases} \frac{u}{(1+\overline{x})^2}, & \text{if } u \geq 0, \\ +\infty, & \text{if } u < 0. \end{cases} \qquad f_q^{\infty}(u) = \begin{cases} u, & \text{if } u \geq 0, \\ +\infty, & \text{if } u < 0. \end{cases}$$

Then $f^{\infty}(1) = f^{i\infty}(1) < f^{\infty}(1; \overline{x}) < f_q^{\infty}(1)$ for all $\overline{x} > 0$.

The following example shows that $f_q^\infty$ seems to provide a finer estimate for the behavior of $f$ at infinity than $f^{i\infty}$ in the quasiconvex case. Also, it shows that we may have $f^\infty \not\equiv f^{i\infty}$.

*Example 2.2* Consider the quasiconvex, proper and lsc function given by

$$f(x) := \begin{cases} +\infty, & \text{if } x < 0, \\ 0, & \text{if } 0 \le x \le 1, \\ 2^k, & \text{if } 2^k < x \le 2^{k+1}, \quad k = 0, 1, 2, \ldots. \end{cases}$$

It is easy to see that

$$f^\infty(u) = \begin{cases} +\infty, & \text{if } u < 0, \\ \frac{u}{2}, & \text{if } u \ge 0. \end{cases}$$

We calculate $f^{i\infty}(1)$. For every $t_k \to +\infty$, let $x_k$ be such that $\frac{x_k}{t_k} \to 1$. Then

$$\limsup_{k\to+\infty} \frac{f(x_k)}{t_k} = \limsup_{k\to+\infty} \frac{f(x_k)}{x_k}\frac{x_k}{t_k} = \limsup_{k\to+\infty} \frac{f(x_k)}{x_k}.$$

Since by construction of $f$ we have $f(x) \le x$ for all $x \ge 0$, using (14) we infer that $f(1) \le 1$. Now, given $\varepsilon > 0$, take $t_k = (1+\varepsilon)2^k$. For every $x_k$ such that $\frac{x_k}{t_k} \to 1$, we have that $\frac{x_k}{t_k} > \frac{1}{1+\varepsilon}$ for large $k$, thus $x_k > \frac{t_k}{1+\varepsilon} = 2^k$. Thus, $f(x_k) \ge 2^k$ and using (14),

$$f^{i\infty}(1) \ge \inf_{\frac{x_k}{t_k}\to 1} \limsup_{k\to+\infty} \frac{f(x_k)}{t_k} \ge \inf_{\frac{x_k}{t_k}\to 1} \limsup_{k\to+\infty} \frac{2^k}{(1+\varepsilon)2^k} = \frac{1}{1+\varepsilon}.$$

Since this is true for all $\varepsilon > 0$, we get $f^{i\infty}(1) = 1$. As for $f_q^\infty(1)$, we have

$$f_q^\infty(1) = \sup_{\substack{x\in \mathrm{dom}\ f \\ t>0}} \frac{f(x+t) - f(x)}{t} \ge \sup_{\substack{x=2^k \\ k\in\mathbb{N}}} \frac{f(x+1) - f(x)}{1} = +\infty.$$

Thus, for this function, $f^\infty(u) < f^{i\infty}(u) < f_q^\infty(u)$ for all $u > 0$.

We now introduce some notions on second order asymptotic functions suitable for dealing with quasiconvex functions.

Given any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and $u \in \mathbb{R}^n$, $u \neq 0$, such that $f^\infty(u)$ is finite, motivated by (7), define

$$f_{qi}^{\infty 2}(u;v) := \sup_{\substack{x \in \mathrm{ri}\,\mathrm{dom} f\\ s>0}} \inf_{t>0} \frac{f(x+tu+sv) - tf^\infty(u) - f(x)}{s}, \ \forall\, v \in \mathbb{R}^n,$$

and the set

$$\widetilde{R}_{qi} := \{u \in \mathbb{R}^n : \ f^\infty(u) = 0, \ f_{qi}^{\infty 2}(u;u) = 0\}.$$

Observe immediately that under convexity of $f$, it holds

$$f_{qi}^{\infty 2}(u;v) = f^{\infty 2}(u;v), \quad v \in \mathbb{R}^n.$$

We start by establishing a simple but important fact.

**Proposition 2.10** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be any proper function and let $\mu := \inf_{\mathbb{R}^n} f$. If $\mu$ is finite, then*

$$[u \in (\mathrm{dom}\ f)^\infty, \ f^\infty(u) = 0] \Longrightarrow f_{qi}^{\infty 2}(u;u) \geq 0.$$

*Proof* Take any $u \in (\mathrm{dom}\ f)^\infty$ such that $f^\infty(u) = 0$. Then

$$\frac{f(x+(s+t)u) - f(x)}{s} \geq \frac{\mu - f(x)}{s}, \ \forall\, x \in \mathrm{dom}\ f, \ \forall\, s,t > 0.$$

This implies that

$$f_{qi}^{\infty 2}(u,u) = \sup_{\substack{x \in \mathrm{ri}\,\mathrm{dom} f\\ s>0}} \inf_{t>0} \frac{f(x+(s+t)u) - f(x)}{s} \geq \sup_{\substack{x \in \mathrm{ri}\,\mathrm{dom} f\\ s>0}} \frac{\mu - f(x)}{s} = 0.$$

Thus $f_{qi}^{\infty 2}(u;u) \geq 0$. □

## 3 Some Applications

This section is devoted to show some potential applications of the notions introduced in the previous sections. The first application concerns the closedness of images via a quasiconvex vector function; the second characterizes the boundedness and nonemptiness of the set of minimizers of a quasiconvex function; whereas the third application deals with the characterization of the boundedness from below along lines of any proper lower semicontinuous function.

### 3.1 Closedness of Images via Quasiconvex Vector Functions

A general result in the convex case was established in [17]; see also [18]; closedness of images via linear mappings may be found in [1] and [19, Theorem 3.10]. Another result is given in [19, Exercise 3.16] for nonlinear mappings. In spite of all these existing results, we will provide an example where our Theorem 3.1 applies and no result previously mentioned does.

We introduce the following class of functions which include those functions that are convex or coercive.

**Definition 3.1** A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ it is said to be in $\mathcal{C}$ if for all $x \in \mathrm{dom}\ f$ and $u \in (\mathrm{dom}\ f)^\infty$, the function $s \mapsto f(x + su)$, $s > 0$, is either unbounded from above or non-increasing.

We recall that, every semistrictly quasiconvex and lower semicontinuous function $f$ is quasiconvex, and that $f_q^\infty$ is quasiconvex whenever $f$ is quasiconvex.

**Theorem 3.1** *Let $F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, be a vector function with each $f_j$, $j = 1, 2, \ldots, m$, being a finite-valued continuous, semistrictly quasiconvex function belonging to $\mathcal{C}$, and let $K \subseteq \mathbb{R}^n$ be closed and convex. Assume that*

$$L_j \subseteq -L_j, \quad \forall\, j \in \{1, 2, \ldots, m\}, \tag{16}$$

*where $L_j := \{u \in K^\infty : \ (f_j)_q^\infty(u) \leq 0\}$. Then $F(K)$ is closed.*

*Proof* By assumption and the remark above, each $L_j$ is a subspace, and therefore, the set $L := \bigcap_{j=1}^m L_j$ is a subspace as well. Let $x_k \in K$ be such that $F(x_k) \to z$, that is, $f_j(x_k) \to z_j$ for all $j = 1, 2, \ldots, m$. Every $x \in \mathbb{R}^n$ can be decomposed as $x = y + y^-$, with $y \in L$ and $y^- \in L^-$, where the set $L^- := \{u \in \mathbb{R}^n : \ \langle x, u\rangle = 0, \ \forall\, x \in L\}$. Thus, $x_k = y_k + y_k^-$.

Since $y_k \in L_j \subseteq -L_j$ for all $j \in \{1, 2, \ldots, m\}$, we obtain $x_k - y_k = y_k^- \in K$. From [15, Lemma 5.5 (a)], it follows that, for all $j = 1, 2, \ldots, m$,

$$f_j(y_k^-) = f_j(x_k - y_k) = f_j(x_k). \tag{17}$$

Suppose that $\sup_k \|y_k^-\| = +\infty$. Then, up to a subsequence, we may assume that $\|y_k^-\| \to +\infty$ and $\frac{y_k^-}{\|y_k^-\|} \to u$. Thus, $u \in K^\infty \cap L^-$ and $\|u\| = 1$. In particular, $u \notin L$ since otherwise $u \in L \cap L^-$ would imply $u = 0$. Hence $(f_j)_q^\infty(u) > 0$ for some $j \in \{1, 2, \ldots, m\}$, so there exist $s > 0$ and $x_0 \in \mathrm{dom}\, f_j$

such that $f_j(x_0+su) > f_j(x_0)$. Due to $f_j \in \mathcal{C}$, $f_j(x_0+su) \to +\infty$ as $s \to +\infty$. This implies that, for $s$ sufficiently large,

$$f_j(x_0 + su) > \max\{f_j(x_0), z_j\}.$$

By quasiconvexity,

$$f_j\left((1 - \frac{s}{\|y_k^-\|})x_0 + \frac{s}{\|y_k^-\|}y_k^-\right) \leq \max\{f_j(x_0), f_j(y_k^-)\},$$

which, by continuity, yields $f_j(x_0 + su) \leq \max\{f_j(x_0), z_j\}$, because of (17), reaching a contradiction.

Hence $\sup_k \|y_k^-\| < +\infty$. Then, up to a subsequence again, we may assume that $y_k^- \to \bar{x} \in K$. It turns out that $f_j(y_k^-) \to f_j(\bar{x})$ for all $j \in \{1, 2, \ldots, m\}$, and therefore $F(x_k) = F(y_k^-) \to F(\bar{x})$, implying that $z = F(\bar{x}) \in F(K)$. $\square$

The function $f(x_1, x_2) = x_1^3$ and $K = [-1, 1] \times \mathbb{R}$ applies to our previous theorem but no result in literature does.

## 3.2 Characterizing Boundedness and Nonemptiness of the Optimal Solution Set

The study of the minimization problem under asymptotic analysis has a long history, for a great account we refer to [1,10]; see also [20], and references therein. None of those works characterizes the nonemptiness and boundednes of the set of minimizers under quasiconvexity. This subsection will provide a result in that direction by means of the first and second order asymptotic functions introduced in Subsection 2.2. Such a result is desirable for algorithmic purposes. Although we only deal with the unconstrained minimization problem, it is rather standard to consider afterwards the constrained problem.

Next two theorems go beyond coerciveness. The continuity of $f$ on dom $f$ and lsc (on $\mathbb{R}^n$) serve to ensure that whenever $x \in \mathrm{ri}\,\mathrm{dom}\, f$ and $u \in (\mathrm{dom}\ f)^\infty$, we have $x + tu \in \mathrm{dom}\ f$ for all $t > 0$, albeit dom $f$ is not necessarily closed, as we can see in the proof.

**Theorem 3.2** *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be proper, continuous on* dom $f$*, lsc* (*on* $\mathbb{R}^n$) *and quasiconvex. Then* $\operatorname*{argmin}_{\mathbb{R}^n} f \neq \emptyset$ *and compact, if and only if the following assertions hold:*

($a$) $f^\infty(u) \geq 0,\ \forall\ u \in \mathbb{R}^n \setminus \{0\}$*;*

($b$) $[u \in (\mathrm{dom}\ f)^\infty,\ f^\infty(u) = 0] \Longrightarrow f_{qi}^{\infty 2}(u; u) \geq 0$;

($c$) $\widetilde{R}_{qi} = \{0\}$.

*Proof* Suppose first that $\operatorname*{argmin}_{\mathbb{R}^n} f \neq \emptyset$ and compact. Obviously ($a$) holds and ($b$) follows from Proposition 2.10. Let $u \in \widetilde{R}_{qi}$, that is, $f_{qi}^{\infty 2}(u; u) = 0$ and $f^\infty(u) = 0$. Then

$$\sup_{\substack{x \in \mathrm{ri}\,\mathrm{dom} f \\ s>0}} \inf_{t>0} \frac{f(x + (s + t)u) - f(x)}{s} = 0,$$

which implies that

$$\inf_{t>0} f(x + (s + t)u) \leq f(x),\ \forall\ x \in \mathrm{ri}\,\mathrm{dom}\ f,\ \forall\ s > 0.$$

Let us prove that for every $x \in \mathrm{ri}\,\mathrm{dom}\ f$ and $s > 0$, $f(x + su) \leq f(x)$. Assume that for some $s > 0$ we have $f(x + su) > f(x)$. Since

$$\inf_{t>0} f(x + (s + t)u) \leq f(x) < f(x + su),$$

there exists $t > 0$ such that $f(x + su + tu) < f(x + su)$. This contradicts the quasiconvexity of $f$ since $x + su$ belongs to the segment $]x, x + su + tu[$.

Take any $x_0 \in \operatorname*{argmin}_{\mathbb{R}^n} f$. Then, there exists a sequence $x_k \in \operatorname{ri}\operatorname{dom} f$ such that $x_k \to x_0$. By continuity of $f$ on dom $f$, $f(x_k) \to f(x_0)$. By lower semicontinuity of $f$,

$$f(x_0 + su) \leq \liminf_{k\to+\infty} f(x_k + su) \leq \lim_{k\to+\infty} f(x_k) = f(x_0).$$

Thus, $x_0 + su \in \operatorname*{argmin}_{\mathbb{R}^n} f$ for all $s > 0$. This contradicts the boundedness of $\operatorname*{argmin}_{\mathbb{R}^n} f$ if $u \neq 0$.

Let us check the reverse implication. Take any minimizing sequence $\{x_k\}$. We will check that it is bounded. Thus, suppose that $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to u$, $u \neq 0$. Since $f(x_k)$ is a bounded (from above) sequence, $f^\infty(u) \leq 0$, and so $f^\infty(u) = 0$. Let $x \in \operatorname{ri}\operatorname{dom} f$. By quasiconvexity, given any $t > 0, s > 0$, we have, for all $k$ sufficiently large,

$$f\left((1 - \frac{(s+t)}{\|x_k\|})x + \frac{(s+t)}{\|x_k\|}x_k\right) \leq \max\{f(x), f(x_k)\}.$$

Given that $\lim_{k\to+\infty} \max\{f(x), f(x_k)\} = \max\{f(x), \inf f\} = f(x)$, the lower semicontinuity of $f$ gives

$$f(x + (s+t)u) \leq \liminf_{k\to+\infty} f\left((1 - \frac{(s+t)}{\|x_k\|})x + \frac{(s+t)}{\|x_k\|}x_k\right) \leq f(x),$$

which implies that $f(x + (s+t)u) - f(x) \leq 0$. Thus $f^{\infty 2}_{qi}(u; u) \leq 0$, which together with assumption $(b)$ give $u \in \widetilde{R}_{qi}$, yielding a contradiction. Hence $\{x_k\}$ is bounded, and so standard arguments show that any limit point is a minimizer for $f$. The same reasoning also proves that $\operatorname*{argmin}_{\mathbb{R}^n} f$ is bounded, so compact. □

The condition that $f$ is lsc (on $\mathbb{R}^n$) is necessary as can be seen by the following example.

*Example 3.1* Take the function $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x_1, x_2) := \begin{cases} x_1^2, & \text{if } x_1 > 0,\ x_2 > 0, \\ 0, & \text{if } x_1 = x_2 = 0, \\ +\infty, & \text{if otherwise.} \end{cases}$$

$f$ is convex, continuous on dom $f$ and $\mathrm{argmin}_{\mathbb{R}^n} f = \{(0,0)\}$. It can be easily seen that $f$ is not lsc on $\mathbb{R}^2$ (for instance at $(0,1)$) and $(c)$ does not hold since for $u = (0,1)$, $f^\infty(u) = f^{\infty 2}_{qi}(u;u) = 0$.

The continuity assumption can be deleted in the preceding theorem at the cost of strengthening the definition of $f^{\infty 2}_{qi}$ and so the set $\widetilde{R}_{qi}$. As before, given $u \in \mathbb{R}^n$, $u \neq 0$, with $f^\infty(u)$ being finite, define

$$f^{\infty 2}_{q}(u;v) := \sup_{\substack{x \in \mathrm{dom}\ f \\ s>0}} \inf_{t>0} \frac{f(x + tu + sv) - tf^\infty(u) - f(x)}{s}, \ \forall\, v \in \mathbb{R}^n,$$

and the set

$$\widetilde{R}_q := \{u \in \mathbb{R}^n :\ f^\infty(u) = 0,\ f^{\infty 2}_{q}(u;u) = 0\}.$$

**Theorem 3.3** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc and quasiconvex function. Then $\underset{\mathbb{R}^n}{\mathrm{argmin}}\ f \neq \emptyset$ and compact, if and only if the following assertions hold:*

$(a)$ $f^\infty(u) \geq 0$, $\forall\ u \in \mathbb{R}^n \setminus \{0\}$*;*

$(b)$ $[u \in (\mathrm{dom}\ f)^\infty,\ f^\infty(u) = 0] \Longrightarrow f^{\infty 2}_{q}(u;u) \geq 0$;

$(c)$ $\widetilde{R}_q = \{0\}$.

*Proof* Suppose first that $\operatorname*{argmin}_{\mathbb{R}^n} f \neq \emptyset$ and compact. Obviously $(a)$ holds and $(b)$ follows from the previous theorem and the fact that $f_q^{\infty 2} \geq f_{qi}^{\infty 2}$. Let $u \in \widetilde{R}_q$, that is, $f_q^{\infty 2}(u;u) = 0$ and $f^\infty(u) = 0$. Then

$$\sup_{\substack{x \in \mathrm{dom}\ f \\ s>0}} \inf_{t>0} \frac{f(x+(s+t)u) - f(x)}{s} = 0,$$

which implies that

$$\inf_{t>0} f(x+(s+t)u) \leq f(x), \quad \forall\ x \in \mathrm{dom}\ f, \quad \forall\ s > 0.$$

In particular, if $\bar{x} \in \operatorname*{argmin}_{\mathbb{R}^n} f$, we get

$$\inf_{t>0} f(\bar{x}+(s+t)u) = f(\bar{x}), \quad \forall\ s > 0. \tag{18}$$

We claim that

$$\bar{x} + (s+t)u \in \operatorname*{argmin}_{\mathbb{R}^n} f, \ \forall\ t > 0, \ \forall\ s > 0.$$

In fact, suppose to the contrary, there exists $r_0 > 0$ such that $f(\bar{x}+r_0 u) > f(\bar{x})$. By quasiconvexity

$$f(\bar{x}+r_0 u) \leq \max\{f(\bar{x}), f(\bar{x}+r_0 u + tu)\} \leq f(\bar{x}+r_0 u+tu), \ \forall\ t > 0,$$

which implies that $f(\bar{x}+r_0 u) \leq \inf_{t>0} f(\bar{x}+r_0 u+tu) = f(\bar{x})$ by (18), and so $\bar{x}+(s+t)u \in \operatorname*{argmin}_{\mathbb{R}^n} f$ for all $t > 0$ and all $s > 0$, yielding a contradiction. Hence $u = 0$.

For the other implication, we proceed as in the previous proof with obvious changes. □

The next example shows that in fact the last two previous theorems cover situations where the function may be non coercive.

*Example 3.2* Let us consider the non coercive quasiconvex function and its respectively asymptotic function

$$f(x) := \begin{cases} -x, & \text{if} \quad x < 0, \\ \dfrac{x}{1+x}, & \text{if} \quad x \geq 0. \end{cases} \qquad f^\infty(u) = \begin{cases} -u, & \text{if} \quad u < 0, \\ 0, & \text{if} \quad u \geq 0. \end{cases}$$

Moreover, if $u > 0$, we get

$$\sup_{\substack{x>0 \\ s>0}} \inf_{t>0} \frac{f(x+su+tu) - tf^\infty(u) - f(x)}{s} = u.$$

Thus, $f^{\infty 2}_{qi}(u;u) = f^{\infty 2}_{q}(u;u) \geq u$, for all $u > 0$. Hence $\widetilde{R}_{qi} = \widetilde{R}_q = \{0\}$.

One may wonder whether

$$f^{\infty 2}_{qi}(u;u) = f^{\infty 2}_{q}(u;u).$$

First of all, we note that

$$f^{\infty 2}_{qi}(u;v) \leq f^{\infty 2}_{q}(u;v),\ \forall\ v \in \mathbb{R}^n,$$

and that if dom $f = \mathbb{R}^n$, then the equality is trivially satisfied. The following instance shows that a strict inequality may hold in general. Define $f$ on $\mathbb{R}^2$ by

$$f(x_1, x_2) := \begin{cases} \frac{\pi}{2}, & \text{if } x_2 > 0, \\ \arctan x_1, & \text{if } x_2 = 0, \\ +\infty, & \text{if otherwise.} \end{cases}$$

Then $f$ is lsc and quasiconvex. Take $u = (1,0) \in (\text{dom } f)^\infty$. Thus for every $x = (x_1, x_2) \in \operatorname{int} \operatorname{dom} f,\ s, t > 0$,

$$\frac{f(x+su+tu) - f(x)}{s} = 0,$$

so $f_{qi}^{\infty 2}(u;u) = 0$. But for $x = (0,0)$, $s > 0$,

$$\inf_{t>0} \frac{f(x + su + tu) - f(x)}{s} > 0,$$

so $f_q^{\infty 2}(u;u) > 0$.

### 3.3 Characterizing Boundedness from Below Along Lines of lsc Functions

The second order asymptotic function provides a better description at infinity than the first order asymptotic function, especially when the function is not coercive or is unbounded from below. In this section, we give a characterization for the boundedness from below, along lines, of any proper lsc function in terms of its first and second order asymptotic functions.

We start by recalling the necessary condition given in [3].

**Theorem 3.4** *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is bounded from below, then the following assertions hold:*

$(a)$ $f^{\infty}(u) \geq 0$ *for all* $u \in (\text{dom } f)^{\infty}$.

$(b)$ $[u \in (\text{dom } f)^{\infty},\ u \neq 0,\ f^{\infty}(u) = 0] \Rightarrow f^{\infty 2}(u;v) \geq 0,\ \forall\ v \in (\text{dom } f)^{\infty 2}[u]$.

If $f$ is proper, convex and $n = 1$, then the converse is also true. More generally, let $f$ be proper, convex and satisfy $(a)$ and the following assumption, which is weaker than $(b)$:

$(b')$ $[u \in (\text{dom } f)^{\infty},\ u \neq 0,\ f^{\infty}(u) = 0] \Rightarrow f^{\infty 2}(u;u) \geq 0$.

Then the restriction of $f$ to every straight line $l = \{x + tu : t \in \mathbb{R}\}$ with $x \in \operatorname{ri} \text{dom } f$, is bounded from below. Indeed, if $f^{\infty}(u) > 0$, then from (2) we get $\lim_{t\to+\infty} f(x+tu) = +\infty$. If $f^\infty(u) = 0$, then by [4, Proposition 4.13] and [4, Remark 4.14], $\lim_{t\to+\infty} f(x+tu)$ is finite. The same is true for $\lim_{t\to-\infty} f(x+tu) = \lim_{t\to+\infty} f(x+t(-u))$, thus $f$ has a lower bound on $l$.

With a little more effort, one can show a similar result for every lsc (not necessarily convex) proper function $f$:

**Proposition 3.1** *For every lsc proper function $f$, conditions $(a)$ and $(b')$ imply that $f$ is bounded from below on every straight line $l = \{x+tu : t \in \mathbb{R}\}$ with $x \in \operatorname{ri}\operatorname{dom} f$.*

*Proof* Assume to the contrary that there exists a sequence $x_n = x + \alpha_n u$ in $l$ such that $f(x_n) \to -\infty$. If the sequence is bounded, we can assume that it converges to some $y \in l$ and obtain that $f(y) = -\infty$, a contradiction. If it is unbounded, by selecting a subsequence and using $-u$ instead of $u$ if necessary, we may assume that $\alpha_n \to +\infty$. Note that, $\frac{x_n}{\alpha_n} \to u$, so

$$0 \geq \liminf_{n\to+\infty} \frac{f(x_n)}{\alpha_n} \geq f^\infty(u) \geq 0.$$

Thus $f^\infty(u) = 0$, and by selecting again a subsequence we may assume that $\frac{f(x_n)}{\alpha_n} \to 0$. Set $s_n = \sqrt{-f(x_n)}$ and $t_n = \frac{\alpha_n}{s_n} - 1$. Then $s_n \to +\infty$, $\lim_{n\to+\infty} t_n \geq \lim_{n\to+\infty}(\frac{\alpha_n}{-f(x_n)} - 1) = +\infty$, and $\frac{x_n}{s_n} - t_n u \to u$. Also,

$$f^{\infty 2}(u;u) \leq \lim_{n\to+\infty} \left( \frac{f(x_n)}{s_n} - t_n f^\infty(u) \right) = -\lim_{n\to+\infty} \sqrt{-f(x_n)} = -\infty.$$

This contradicts $f^{\infty 2}(u;u) = 0$. $\square$

Proposition 3.1 does not imply that a lsc proper function that satisfies conditions $(a)$ and $(b')$, is bounded from below. In the following counterexample, the function $f$ is also convex and satisfies $(a)$ and $(b)$.

*Example 3.3* Define $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ by

$$f(\alpha, \beta) := \begin{cases} -\sqrt{\beta}, & \text{if } \alpha \geq 0 \text{ and } \beta \in [0, \sqrt{\alpha}], \\ \\ +\infty, & \text{if otherwise.} \end{cases}$$

Note that $f^\infty(1,0) = 0$ and $f^\infty(\alpha,\beta) = +\infty$ for all $(\alpha,\beta) \notin \mathbb{R}_+(1,0)$. Also, $f^{\infty 2}((1,0);(1,0)) = 0$. Let $v = (\alpha,\beta)$ with $\alpha \in \mathbb{R}$ and $\beta \geq 0$. Choose a point $(x_1,x_2) \in \operatorname{ri}\operatorname{dom} f$. For every $s > 0$, and for $t$ sufficiently large, we have that $(x_1,x_2) + t(1,0) + s(\alpha,\beta)$ belongs to the domain of $f$. Clearly

$$\begin{aligned} f^{\infty 2}((1,0);(\alpha,\beta)) &= \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{f((x_1,x_2)+t(1,0)+s(\alpha,\beta)) - tf^\infty(1,0)}{s} \\ &= \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{-\sqrt{x_2+s\beta}}{s} = 0 \end{aligned}$$

Suppose $\beta < 0$, then for $s$ sufficiently large one has $x_2 + s\beta < 0$, thus $(x_1,x_2) + t(1,0) + s(\alpha,\beta) \notin \operatorname{dom} f$ for all $t > 0$. Hence, in this case, we have $f^{\infty 2}((1,0);(\alpha,\beta)) = +\infty$. Then, $f$ satisfies conditions $(a)$ and $(b)$. In particular, the restriction of $f$ on any straight line is bounded from below. However, it is not bounded from below on $\mathbb{R}^2$ since $\lim_{x\to+\infty} f(x,\sqrt{x}) = -\infty$.

## 4 Conclusions

In this paper, we developed first and second order asymptotic analysis suitable for dealing with quasiconvex optimization problems. In particular, several notions of asymptotic functions, starting from those of asymptotic cones, are studied and compared between them. For an appropriate asymptotic function, we gave a formula via the two Dini directional derivatives. In addition, we also introduced notions of second order asymptotic functions in the quasiconvex case, based on corresponding first order notions. Applications of Section 3 show the importance of the studied generalized asymptotic analysis under generalized convexity assumptions. In this way, connections between these generalized asymptotic functions with other areas of the generalized convexity theory, such as: generalized conjugacies, subdifferentials and support functions, are expected. One problem that is left open, is whether in higher dimension, the usual asymptotic function inherits quasiconvexity.

One might argue that the term "second order asymptotic cone" is not appropriate. A referee suggested "secondary asymptotic cone" or "higher-order asymptotic cone". However, for the moment we keep this terminology until more convincing arguments arise: there is much more to be understood about our notion. The definite term must reflect the real phenomenon searched.

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