Exceptional Families of Elements for Variational Inequalities in Banach Spaces

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Abstract. In keeping with very recent efforts to establish a useful concept of an "exceptional family of elements" for variational inequality problems rather than complementarity problems as in the past we propose such a concept. It generalizes previous ones to multivalued variational inequalities in general normed spaces and allows us to obtain several existence results for variational inequalities corresponding to earlier ones for complementarity problems. Compared with the existing literature, we consider problems in more general spaces and under considerably weaker assumptions on the defining map.

Key Words. Variational inequalities, complementarity problems, quasi-monotone maps, exceptional families of elements.

1 Introduction

In the study of complementarity problems (CP) and the more general variational inequality problems (VIP) considerable emphasis has been placed on the question of existence of solutions. Necessary as well as sufficient conditions have been derived both for bounded and unbounded domains using different approaches. Some authors deal with the unbounded case using the same assumptions as in the bounded case together with a suitable coercivity condition necessary to obtain the nonemptiness of the solution set. Other authors use the concept of an "exceptional family of elements". Several such concepts have been proposed depending on the particular assumptions on the problem under investigation. Results in this stream of literature usually take on the form: if a problem does not have a solution, then there exists an exceptional family of elements. This immediately leads to the following existence result: if an exceptional family of elements does not exist, then the problem has a solution. This second approach usually involves a strong continuity assumption of the map.

Until very recently exceptional families of elements were studied in the context of CP as initially (Ref. 1). Meanwhile some efforts have been made to introduce similar concepts for VIP. The definition depends on the particular assumptions on the VIP. The present article proposes a new concept of an exceptional family of elements under rather general assumptions on the VIP. With help of it we are able to derive several existence results for multivalued VIP in a general normed space for maps defining the VIP which are considerably more general than those studied before in the literature. Indeed we can prove our existence result without making use of strong continuity assumptions on the map, but under the same assumptions as for a bounded domain.

In Section 2 we provide the necessary background from the literature before we generalize in Section 3 some of the existing concepts of an exceptional family of elements for a rather large class of VIP. Here we obtain our main results. Section 4 demonstrates how the results in Section 3 can be used to derive existence results for multivalued VIP in normed spaces with and without generalized monotonicity assumptions.

2 Background

In this section we review some of the earlier results related to our study. Let a normed space X, a convex subset K of X and a multivalued map $T : K \rightrightarrows X^*$ be given, where X^* denotes the topological dual of X. We recall that the Variational Inequality Problem (VIP) is the following:

$$\text{find } x_0 \in K : \exists x_0^* \in T(x_0) \text{ such that } \forall x \in K,\ \langle x_0^*, x - x_0 \rangle \geq 0. \tag{VIP}$$

Note that this is the so-called strong VIP, and it is the only one we consider in this paper. Whenever K is a convex cone, we get as a particular case the Complementarity Problem (CP):

$$\text{find } x_0 \in K : \exists x_0^* \in T(x_0) \text{ such that } x_0^* \in K^* \text{ and } \langle x_0^*, x_0 \rangle = 0. \tag{CP}$$

Here $K^* = \{x^* \in X^* : \langle x^*, x \rangle \geq 0, \forall x \in K\}$ is the dual cone of K.

In recent years several papers were devoted to the study of the complementarity problem through the use of "exceptional families of elements".

These were defined in the special case of a Hilbert space and for a multivalued map as follows (Ref. 2).

Definition 2.1 Given a convex cone K in a Hilbert space H and a map $T : K \rightrightarrows H$, a family $\{x_r\}_{r>0} \subset K$ is an exceptional family of elements for T, if for every $r > 0$ there exist a real number $\mu_r > 0$ and an element $x_r^* \in T(x_r)$ such that the following conditions are satisfied:

(i) $u_r := \mu_r x_r + x_r^* \in K^*$,

(ii) $\langle u_r, x_r \rangle = 0$,

(iii) $\|x_r\| \to \infty$ as $r \to +\infty$.

This notion was first introduced in the form of an exceptional sequence of elements by Smith (Ref. 1) for the particular case where T is a single-valued map and $H = \mathbb{R}^n$. Later on this notion was explored mainly by Isac and many others who generalized it to multivalued maps and to Hilbert spaces; e.g., Refs. 2–5 . In these papers T is assumed to be a "completely upper semicontinuous field". This means that T has a representation of the form $T = x - h(x)$ where $h : H \to H$ is a completely continuous mapping, namely for any bounded set $A \subseteq H$, the image of A through the map $h(x) = x - T(x)$ is relatively compact. The following result (Theorem 4.1 in Ref. 2) is characteristic of the results of the above papers.

Theorem 2.1 Let H be a Hilbert space, $K \subseteq H$ be a pointed closed convex cone and $T : H \rightrightarrows H$ be a multivalued, completely upper semicontinuous field, with nonempty compact contractible values. If there does not exist a solution of CP, then there exists an exceptional family of elements for T.

Theorems of this kind were proven valuable for showing the existence of solutions of complementarity problems; indeed, for a map satisfying the assumptions of the theorem, in order to show the existence of a solution for CP it is sufficient to show that there does not exist an exceptional family of elements. However complete upper semicontinuity is a stringent restriction in infinite dimensional Hilbert spaces. For instance a constant map is not a completely upper semicontinuous field. The necessity for such an assumption is due to the fact that tools such as degree theory are used.

Generalizations of the above theorem by weakening the assumptions have followed various directions. In Refs. 6–9 exceptional families were generalized to variational inequality problems with single-valued completely continuous fields defined in $\mathbb{R}^n$ or more generally in a Hilbert space. Many definitions have been proposed; we recall the definition of exceptional families as introduced in Ref. 9 .

Definition 2.2 Given an unbounded convex subset K of a Hilbert space H, a map $T : K \to H$ and a point $\hat{x} \in H$, a family $\{x_r\}_{r>0} \subset K$ is an exceptional family of elements for T with respect to $\hat{x}$, if for every r sufficiently large there exists a real number $\mu_r > 0$ such that the following conditions are satisfied:

(i) $\|x_r\| \to \infty$ as $r \to +\infty$,

(ii) $T(x_r) + \mu_r(x_r - \hat{x}) \in -N_K(x_r)$.

Here $N_K(x_r)$ is the normal cone to K at the point x_r; i.e.,

$$N_K(x_r) = \{x^* \in K : \langle x^*, x - x_r \rangle \leq 0, \forall x \in K\} .$$

It is easy to see that in the case of single-valued maps this definition generalizes the corresponding definition for complementarity problems where

K is a cone and $\hat{x} = 0$ (see Definition 2.1).

Variational inequalities in Hilbert spaces with multivalued, completely upper semicontinuous fields were considered in Ref. 7 . But there the definition of an exceptional family is different. Relation (ii) is replaced by another one where T and N_K are calculated at different points. Very recently in Ref. 10 exceptional families for single-valued complementarity problems were considered for the first time in Banach spaces. These spaces are assumed to be uniformly smooth and uniformly convex. Finally in what concerns the assumption that T is a completely upper semicontinuous field, for the case of a complementarity problem it was replaced by the weaker assumption that T is "regularly completely continuous" in Ref. 11 (in case of a single-valued map T) and by a very weak continuity assumption in Ref. 12 .

The main result of this paper generalizes the above results in several directions. We will define the notion of an exceptional family of elements for a multivalued variational inequality problem in a general normed space and derive a theorem from which results like Theorem 2.1 follow. No further assumption on the space will be made, and the map T need not be a completely continuous field. Also, the map is not necessarily defined on the whole space as it is the case in all papers mentioned on the subject except for Refs. 11 and 12.

3 Main Results

In preparation of the new definition of an exceptional family we introduce the following notation. Given a set A and a point x in a normed space X, $dist(x, A)$ will be the distance from x to A, $\mathbb{R}_+ A = \{tx : t \geq 0 \text{ and } x \in A\}$ and $\mathbb{R}_{++} A = \{tx : t > 0 \text{ and } x \in A\}$.

The so-called duality map J is defined as follows:

$$J(x) = \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}.$$

In other words, $J(x)$ contains all elements x^* of X^* with norm equal to $\|x\|$ such that the maximum of $\langle x^*, \cdot \rangle$ on the closed ball $\bar{B}(0, \|x\|)$ is attained at x. In the special case where X^* is a strictly convex normed space, $J(x)$

is a singleton. If in addition X is a Hilbert space H and we identify H with H^*, then we get $J(x) = \{x\}$.

Let $\hat{x} \in X$ be fixed throughout the paper. Given $x \in X$, let $L(x) = N_{\bar{B}(\hat{x},\|\hat{x}-x\|)}(x)\backslash\{0\}$ be the normal cone to the closed ball $\bar{B}(\hat{x}, \|\hat{x} - x\|)$ at x from which we extract 0. The set $L(x)$ is nonempty since $\bar{B}(\hat{x}, \|\hat{x} - x\|)$ has a nonempty interior for all $x \neq \hat{x}$. The multivalued map L can be written in terms of the duality map J of the space X. Obviously the normal cone to $\bar{B}(0, \|x\|)$ at x is $\mathbb{R}_{++}J(x)$. Hence we get by parallel translation

$$L(x) = \mathbb{R}_{++}J(x - \hat{x}). \tag{1}$$

Our definition of an exceptional family of elements runs as follows.

Definition 3.1 Given an unbounded convex subset K of a normed space X, a map $T : K \rightrightarrows X^*$ and a point $\hat{x} \in X$, a family of elements $\{x_r\}_{r>0}$ of K is an exceptional family of elements (EFE) for T with respect to $\hat{x}$ if

(i) $\lim_{r\to+\infty} \|x_r\| = +\infty$ and

(ii) for any $r > r_0 := dist(\hat{x}, K)$ there exists $x_r^* \in T(x_r)$ and $y_r^* \in L(x_r)$

such that $x_r^*+y_r^* \in -N_K\,(x_r)$; in set language, $0 \in T(x_r)+\mathbb{R}_{++}J\,(x_r-\hat{x})+N_K(x_r)$.

Note that condition (ii) means that x_r solves VIP on the whole set K for the map $T(x)+L(x)$. In the special case where X is a Hilbert space $J(x)=\{x\}$, and by relation (1), $L(x)=\{\mu(x-\hat{x}) : \mu>0\}$. Thus, condition (ii) above becomes: there exist $x_r^* \in T(x_r)$ and $\mu_r>0$ such that $x_r^*+\mu_r\,(x_r-\hat{x}) \in -N_K(x_r)$; i.e., we recover Definition 2.2. Hence Definition 3.1 generalizes Definition 2.2 to multivalued maps in normed spaces. If further K is a cone and we take $\hat{x}=0$, then obviously $r_0=0$. In this case condition (ii) means that

$$\forall x \in K, \quad \langle x_r^*+\mu_r x_r, x-x_r\rangle \geq 0.$$

Since K is a cone, it is a standard trick to show that the vector $u_r := x_r^*+\mu_r x_r$ satisfies $\langle u_r, x_r\rangle = 0$ and $u_r \in K^*$; see for instance Ref. 13. Hence, we recover Definition 2.1 of exceptional families for CP. Thus Definition 3.1 also generalizes Definition 2.1 to variational inequality problems in normed spaces.

In view of (i) and (ii) in Definition 3.1 it is sufficient to define an EFE for $r > r_0$ as above. For $0 < r \leq r_0$ it can be defined arbitrarily.

We set $K_r = K \cap \bar{B}(\hat{x}, r)$. The result below relates exceptional families to partial solutions of VIP and it is the key result of this section.

Theorem 3.1 Let K be an unbounded closed convex set in a normed space X, $T : K \rightrightarrows X^*$ be a map and $\hat{x} \in X$ be fixed. Assume that for every $r > r_0$ there exists $x_r \in K_r$ and $x_r^* \in T(x_r)$ such that $\langle x_r^*, x - x_r \rangle \geq 0$ holds for every $x \in K_r$, but not for every $x \in K$. Then $\{x_r\}_{r>r_0}$ is an exceptional family of elements for T with respect to $\hat{x}$.

Proof. First, note that $\|x_r - \hat{x}\| = r$. Indeed, it is known that whenever $\|x_r - \hat{x}\| < r$, then $\langle x_r^*, x - x_r \rangle \geq 0$ holds for every $x \in K$ (see Lemma 3.1 in Ref. 12); but this is excluded by our assumption. It follows that $\lim_{r \to +\infty} \|x_r\| = +\infty$.

Since $\langle -x_r^*, x - x_r \rangle \leq 0$ for all $x \in K_r$, we infer from the definition of N_{K_r} that $x_r^* \in -N_{K_r}(x_r)$. Since $K_r = K \cap \bar{B}(\hat{x}, r)$ and $\bar{B}(\hat{x}, r)$ has a nonempty interior, it is known that $N_{K_r}(x_r) = N_K(x_r) + N_{\bar{B}(\hat{x},r)}(x_r)$ (Ref. 14 , Theorem

4.1.16). Hence there exists $y_r^* \in N_{\bar{B}(\hat{x},r)}(x_r)$ such that $x_r^* + y_r^* \in -N_K(x_r)$. Note that $x_r^* \notin -N_K(x_r)$ since otherwise we would have $\langle x_r^*, x - x_r \rangle \geq 0$ for every $x \in K$, a contradiction. It follows that $y_r^* \neq 0$. Using $\|x_r - \hat{x}\| = r$ we infer that $y_r^* \in L(x_r)$. Consequently, the family $\{x_r\}_{r>r_0}$ is an EFE according to Definition 3.1. ■

Corollary 3.1 Let K be an unbounded closed convex set in a normed space X, $\hat{x} \in X$, and $T : K \rightrightarrows X^*$ be a map such that VIP has a solution in every K_r, $r > dist(\hat{x}, K)$. If there is no solution for VIP in all K, then there exists an exceptional family of elements for T with respect to $\hat{x}$.

Proof. Assume that VIP has no solution in K. For each $r > r_0 = dist(\hat{x}, K)$, let x_r be a solution of VIP in K_r. Then there exists $x_r^* \in T(x_r)$ such that $\langle x_r^*, x - x_r \rangle \geq 0$ for all $x \in K_r$. Since x_r is not a solution of VIP in K, Theorem 3.1 entails that $\{x_r\}_{r>r_0}$ is an exceptional family of elements for T with respect to $\hat{x}$. ■

4 Applications

In this section we derive two implications of Theorem 3.1 and Corollary 3.1 above. The first one is obtained under a generalized monotonicity assumption and a weak continuity assumption. The second result is derived with a strong continuity assumption, without a generalized monotonicity assumption.

We first assume that T is a quasimonotone map. Recall that $T : K \rightrightarrows X^*$ is called quasimonotone if for every $x, y \in K$ and $x^* \in T(x)$, $y^* \in T(y)$, the following implication holds:

$$\langle x^*, y - x\rangle \geq 0 \quad \Longrightarrow \quad \langle y^*, y - x\rangle \geq 0.$$

We impose a very weak kind of continuity (Ref. 15): T is called upper sign-continuous if for all $x, y \in K$ the following implication holds, where $x_t = ty + (1-t)\,x$:

$$\left(\forall t \in (0,1)\,,\ \inf_{x^* \in T(x_t)} \langle x^*, y - x\rangle \geq 0\right) \Longrightarrow \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0.$$

Corollary 4.1 Let K be a closed convex set in a normed space X, $\hat{x} \in X$ and $T : K \rightrightarrows X^*$ be a quasimonotone, upper sign-continuous map with nonempty, w*-compact and convex values. Assume that for every $r > r_0 = dist(\hat{x}, K)$ the set K_r is weakly compact. If there is no solution for VIP in all K, then there exists an exceptional family of elements for T with respect to $\hat{x}$.

Proof. According to Theorem 2.1 in Ref. 16 , the assumptions guarantee that for every $r > r_0$ VIP has a solution in K_r. Hence Corollary 3.1 applies and yields the desired result. ∎

We note that if X is reflexive, the sets K_r are weakly compact as required in the above corollary.

If we do not impose any generalized monotonicity assumption, we get an analogous result, at the expense of a much stronger continuity assumption. As an example, we consider regular completely upper semicontinuous maps, i.e., maps $T : K \rightrightarrows X^*$ with the following two properties:

(**P1**) the image of any bounded subset of K is relatively compact;

(**P2**) if the sequence $x_n \in K$, $n \in \mathbb{N}$ converges weakly to x and the sequence $x_n^* \in T(x_n)$ converges strongly to x^*, then $x^* \in T(x)$.

This is a generalization of the notion of a regular completely continuous map considered in Ref. 11 , adapted to the multivalued case. We note that "regular complete continuity" is the weakest continuity assumption used so far in papers dealing with EFE, with the exception of Ref. 12 where quasi-monotonicity was also imposed. We first show the following lemma.

Lemma 4.1 Let X be a reflexive Banach space, $K \subseteq X$ be a bounded closed convex set and $T : K \rightrightarrows X^*$ be a regular completely upper semicontinuous map with nonempty convex values. Then VIP has a solution.

Proof. For every $x \in K$, the set $T(x)$ is relatively compact by property (P1). Hence for each sequence $(x_n^*)_{n\in\mathbb{N}} \subset T(x)$, there exists a subsequence converging to some element $x^* \in \overline{T(x)}$. Property (P2) ensures that $x^* \in T(x)$. Thus, $T(x)$ is compact.

For each $x \in K$ define the set

$$G(x) = \{y \in K : \exists y^* \in T(y) \text{ such that } \langle y^*, x - y\rangle \geq 0\}.$$

We show that $G(x)$ is compact. Indeed, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in

$G(x)$. Choose $y_n^* \in T(y_n)$ such that $\langle y_n^*, x - y_n \rangle \geq 0$ holds. Since K is bounded, $T(K)$ is relatively compact. Hence there exists a subsequence $\left(y_{n_k}^*\right)_{k\in\mathbb{N}}$ of $(y_n^*)_{n\in\mathbb{N}}$ converging to some element $y^* \in \overline{T(K)}$. Since X is reflexive and $(y_{n_k})_{k\in\mathbb{N}}$ is bounded, there exists a subsequence $\left(y_{n_{k_l}}\right)_{l\in\mathbb{N}}$ of $(y_{n_k})_{k\in\mathbb{N}}$ converging weakly to some $y \in K$. By our assumption on T, $y^* \in T(y)$. By taking the limit as $l \to +\infty$ we infer that $\langle y^*, x - y \rangle \geq 0$, i.e., $y \in G(x)$ and $G(x)$ is compact.

The rest of the proof is a routine application of KKM theory. One first shows that the map G satisfies the KKM property, i.e., for all $x_1, x_2, \ldots x_n$ in K and all $x \in co\{x_1, x_2, \ldots x_n\}$,

$$G(x) \subseteq \bigcup_{i=1}^{n} G(x_i)$$

holds. From the well-known Ky Fan Lemma it follows that $\bigcup\limits_{x\in K} G(x) \neq \emptyset$. Let $x_0 \in \bigcup\limits_{x\in K} G(x)$. Then

$$\forall x \in K, \exists x^* \in T(x_0) \text{ such that } \langle x^*, x - x_0 \rangle \geq 0.$$

This means that $\min_{x\in K} \max_{x^*\in T(x_0)} \langle x^*, x - x_0 \rangle \geq 0$. By applying the minimax theorem we infer that $\max_{x^*\in T(x_0)} \min_{x\in K} \langle x^*, x - x_0 \rangle \geq 0$, hence x_0 is a solution of the variational inequality problem. ■

Corollary 4.2 Let X be a reflexive Banach space, $K \subseteq X$ be an unbounded closed convex set, $\widehat{x} \in X$ and $T : K \rightrightarrows X^*$ be a regular completely upper semicontinuous map with nonempty convex values. If there is no solution for VIP in all K, then there exists an exceptional family of elements for T with respect to $\hat{x}$.

Proof. This is an obvious consequence of Lemma 4.1 and Corollary 3.1. ■

The above Corollary was also shown in Ref. 11 for the particular case where X is a Hilbert space, T is single-valued and K is a cone.

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