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A Quasiconvex Asymptotic Function with Applications in Optimization

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Abstract We introduce a new asymptotic function, which is mainly adapted to quasiconvex functions. We establish several properties and calculus rules for this concept and compare it to previous notions of generalized asymptotic functions. Finally, we apply our new definition to quasiconvex optimization problems: we characterize the boundedness of the function, and the nonemptiness and compactness of the set of minimizers. We also provide a sufficient condition for the closedness of the image of a nonempty closed and convex set via a vector-valued function.

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1 Introduction

The notion of asymptotic cone of an unbounded set has been introduced in order to study its behavior at infinity. The asymptotic cone of the epigraph of a function, which yields its asymptotic function, provides a description of the function at infinity.

Those notions are an outstanding tool for studying problems with unbounded data and have given rise to the branch of mathematics called asymptotic analysis (see [1]). They have been employed for studying optimization problems such as scalar minimization, vector optimization, variational inequalities and equilibrium problems (see [1–6] and references therein).

Clearly, convexity is a simplifying assumption, when studying minimization problems. In convex minimization, any local minimizer is global, first order necessary optimality conditions become also sufficient, and the asymptotic cones of nonempty sublevel sets coincide. For nonconvex functions, none of the above holds in general.

As was noted in [7,8], the usual asymptotic function is not good enough to describe the behavior of a nonconvex function at infinity. In the particular case of quasiconvex functions, many alternative generalized asymptotic functions were given in the last years (see [8–11] and references therein). All those attempts provide a similar characterization of the nonemptiness and compactness of the solution set of the original function, while other properties and calculus rules are lost.

In this paper, we introduce a new notion of asymptotic function to deal with quasiconvexity, which provides information on the value of the original function at infinity. Our definition preserves many properties and calculus rules of the usual asymptotic function, beyond the characterization of the nonemptiness and compactness of the solution set for scalar minimization problems. The new asymptotic function can also be used to study some properties of the original function. However, in contrast to previous definitions, the new asymptotic function is related to the sublevel sets of the function, rather than to its epigraph. This is natural, since we are dealing with quasiconvexity.

The paper is organized as follows. In Section 2, we include notation and preliminaries. We review some standard facts on asymptotic analysis and generalized convexity. In Section 3, we introduce our new asymptotic function for dealing with the quasiconvex case. We also provide several properties, calculus rules and the comparison with previous notions of generalized asymptotic functions. Finally, in Section 4, we apply our new definition to characterize the boundedness (from below and above) of the function, to characterize the nonemptiness and compactness of the set of minimizers, and to provide a sufficient condition to ensure the closedness of the image of a closed and convex set via a quasiconvex vector-valued function. We also provide some examples of evaluation of the new asymptotic function for some fundamental particular cases, such as quadratic functions and fractions of two affine functions.

2 Preliminaries and Basic Definitions

In this paper, we denote the scalar product between two elements of $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$ and the norm by $\|\cdot\|$. For $K \subseteq \mathbb{R}^n$, its closure is denoted by $\operatorname{cl} K$, its boundary by $\operatorname{bd} K$, its topological interior by $\operatorname{int} K$, its relative interior by $\operatorname{ri} K$ and its convex hull by $\operatorname{conv} K$. By K^* we denote the positive polar cone of K. The indicator function of K is defined by $\delta_K(x) := 0$, if $x \in K$, and by $\delta_K(x) := +\infty$ elsewhere. The support function of K is defined by $\sigma_K(y) := \sup_{x \in K} \langle x, y \rangle$. By $B(x, \delta)$ we mean the open ball with center at $x \in \mathbb{R}^n$ and radius $\delta > 0$.

Given any function $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the effective domain of f is defined by $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. We say that f is a proper function if $f(x) > -\infty$ for every $x \in \mathbb{R}^n$ and $\operatorname{dom} f$ is nonempty. For a function f, we adopt the usual convention $\inf_{\emptyset} f := +\infty$ and $\sup_{\emptyset} f := -\infty$.

We denote by $\operatorname{epi} f := \{(x,t) \in \operatorname{dom} f \times \mathbb{R} : \ f(x) \leq t\}$ its epigraph and for a given $\lambda \in \mathbb{R}$ by $S_\lambda(f) := \{x \in \mathbb{R}^n : \ f(x) \leq \lambda\}$ its sublevel set at value λ. As usual, $\operatorname{argmin}_K f := \{x \in K : \ f(x) \leq f(y), \ \forall\, y \in K\}$.

A proper function f is said to be:

(a) semistrictly quasiconvex, if its domain is convex and for every $x, y \in \operatorname{dom} f$ with $f(x) \neq f(y)$,

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}, \ \forall \ \lambda \in]0,1[.$$

(b) quasiconvex, if for every $x, y \in \operatorname{dom} f$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \ \forall \ \lambda \in [0,1].$$

Every convex function is quasiconvex, and every semistrictly quasiconvex and lower semicontinuous (lsc from now on) function is quasiconvex (see [12, Theorem 2.3.2]). The continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := \min\{|x|, 1\}$, is quasiconvex, without being semistrictly quasiconvex.

Recall that

$$f \text{ is convex} \Longleftrightarrow \operatorname{epi} f \text{ is a convex set.}$$
$$f \text{ is quasiconvex} \Longleftrightarrow S_\lambda(f) \text{ is a convex set}, \text{ for all } \lambda \in \mathbb{R}.$$

For a further study on generalized convexity, we refer to [12–14].

As explained in [1], the notions of asymptotic cone and the associated asymptotic function have been employed in optimization theory in order to handle unbounded and/or nonsmooth situations, in particular when standard compactness hypotheses are absent. We recall some basic definitions and properties of asymptotic cones and functions, which can be found in [1].

For a nonempty set $K \subseteq \mathbb{R}^n$ its asymptotic cone is defined by

$$K^\infty := \left\{ u \in \mathbb{R}^n : \exists\, t_k \to +\infty,\ \exists\, x_k \in K,\ \frac{x_k}{t_k} \to u \right\}.$$

We adopt the convention that $\emptyset^\infty = \emptyset$.

When K is a closed and convex set, it is known that the asymptotic cone is equal to (see [1, Proposition 2.1.5])

$$K^\infty = \left\{ u \in \mathbb{R}^n : x_0 + \lambda u \in K,\ \forall\, \lambda \geq 0 \right\} \text{ for any } x_0 \in K. \tag{1}$$

The basic properties of the asymptotic cone are listed below.

Proposition 2.1 *Let $\emptyset \neq K \subseteq \mathbb{R}^n$, then*

(a) *If $K_0 \subseteq K$, then $(K_0)^\infty \subseteq K^\infty$.*

(b) *$(K + x_0)^\infty = K^\infty$ for all $x_0 \in \mathbb{R}^n$.*

(c) *$K^\infty = (\overline{K})^\infty$.*

(d) *$K^\infty = \{0\}$ iff K is bounded.*

(e) *Let $\{K_i\}_{i\in I}$ be a family of sets from $\mathbb{R}^n$. Then $\bigcup_{i\in I}(K_i)^\infty \subseteq (\bigcup_{i\in I} K_i)^\infty$.*

The equality holds when $|I| < +\infty$.

(f) *Let $\{K_i\}_{i\in I}$ be a family of sets from $\mathbb{R}^n$ satisfying $\bigcap_{i\in I} K_i \neq \emptyset$. Then*

$$\left(\bigcap_{i\in I} K_i \right)^\infty \subseteq \bigcap_{i\in I} (K_i)^\infty.$$

The equality holds when every K_i is closed and convex.

The asymptotic function $f^\infty : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ of a proper function f as before, is the function for which

$$\operatorname{epi} f^\infty := (\operatorname{epi} f)^\infty. \tag{2}$$

From this, one may show that

$$f^\infty(u) = \inf \left\{ \liminf_{k \to +\infty} \frac{f(t_k u_k)}{t_k} : \ t_k \to +\infty, \ u_k \to u \right\}. \tag{3}$$

Moreover, when f is lsc and convex, for all $x_0 \in \operatorname{dom} f$ we have

$$f^\infty(u) = \sup_{t>0} \frac{f(x_0 + tu) - f(x_0)}{t} = \lim_{t \to +\infty} \frac{f(x_0 + tu) - f(x_0)}{t}. \tag{4}$$

A function f is called coercive if $f(x) \to +\infty$ as $\|x\| \to +\infty$. If $f^\infty(u) > 0$ for all $u \neq 0$, then f is coercive. In addition, if f is convex and lsc, then (see [1, Proposition 3.1.3])

$$f \text{ is coercive} \Longleftrightarrow f^\infty(u) > 0, \ \forall\, u \neq 0 \Longleftrightarrow \operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset \text{ and compact.} \tag{5}$$

The problem of finding an adequate definition of an asymptotic function has been studied in the last years, since the usual asymptotic function is not well suited for the description of the behavior of a nonconvex function at infinity. Several attempts to deal with the quasiconvex case have been made in [7–10] while applications to optimization can be found in [10, 11].

The following two asymptotic functions to deal with quasiconvexity were introduced in [9]. Recall that, given a proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the q-asymptotic function is defined by

$$f_q^{\infty}(u) := \sup_{x \in \operatorname{dom} f} \sup_{t>0} \frac{f(x+tu) - f(x)}{t}. \tag{6}$$

Given $\lambda \in \mathbb{R}$ with $S_\lambda(f) \neq \emptyset$, the λ-asymptotic function is defined by

$$f^{\infty}(u;\lambda) := \sup_{x \in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu) - \lambda}{t}. \tag{7}$$

If f is lsc and quasiconvex, by [9, Theorem 4.7] we have

$$f_q^{\infty}(u) > 0, \ \forall\ u \neq 0 \iff \operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset \text{ and compact}, \tag{8}$$

and by [9, Proposition 5.3]

$$f^{\infty}(u;\lambda) > 0, \ \forall\ u \neq 0 \iff S_\lambda(f) \neq \emptyset \text{ and compact}. \tag{9}$$

If f is quasiconvex (resp. lsc), then $f^q(\cdot)$ and $f^{\infty}(\cdot;\lambda)$ are quasiconvex (resp. lsc). Furthermore, the following relations hold for any $\lambda \in \mathbb{R}$ with $S_\lambda(f) \neq \emptyset$,

$$f^{\infty} \leq f^{\infty}(\cdot;\lambda) \leq f_q^{\infty}. \tag{10}$$

Both inequalities could be strict even for quasiconvex functions, as was proved in [9, Example 5.6].

Finally, it is important to point out that the fact that $f_q^\infty(u) > 0$ for all $u \neq 0$ does not imply that f is coercive as the function $f(x) := \frac{|x|}{1+|x|}$ shows. Hence, the characterization (8) goes beyond coercivity.

3 A Quasiconvex Asymptotic Function

In this section, we introduce a new definition of an asymptotic function to deal with quasiconvex functions. We establish properties and calculus rules and compare with previous notions of asymptotic function.

3.1 Definition and Properties

The usual definition of the asymptotic function involves the asymptotic cone of the epigraph. This explains why that definition is useful mainly for convex functions. Our definition, quite naturally, involves the asymptotic cone of the sublevel sets of the original function.

Definition 3.1 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. We define the qx-asymptotic function $f^{qx} : \mathbb{R}^n \to \overline{\mathbb{R}}$ of f by

$$f^{qx}(u) := \inf \{\lambda : \; u \in (S_\lambda(f))^\infty\}. \tag{11}$$

Since f is lsc and quasiconvex, $S_\lambda(f)$ is a closed and convex set. For any λ such that $S_\lambda(f) \neq \emptyset$, by Proposition 2.1($f$) we have

$$S_\lambda(f^{qx}) = \bigcap_{\mu>\lambda} (S_\mu(f))^\infty = \left(\bigcap_{\mu>\lambda} S_\mu(f)\right)^\infty = (S_\lambda(f))^\infty. \tag{12}$$

The following remark follows immediately from the previous equation.

Remark 3.1

(i) The first equality in (12) holds for every $\lambda \in \mathbb{R}$ and implies that $S_\lambda(f^{qx})$ is a closed and convex cone. Hence f^{qx} is lsc, quasiconvex, and positively homogeneous of degree 0.

(ii) The qx-asymptotic function is monotone in the sense that $f_1 \le f_2$ implies that $(f_1)^{qx} \le (f_2)^{qx}$. In fact, take $\lambda \in \mathbb{R}$ such that $S_\lambda(f_2) \neq \emptyset$, then

$$S_\lambda(f_2) \subseteq S_\lambda(f_1) \implies (S_\lambda(f_2))^\infty \subseteq (S_\lambda(f_1))^\infty \iff S_\lambda(f_2)^{qx} \subseteq S_\lambda(f_1)^{qx},$$

which means that $(f_1)^{qx} \le (f_2)^{qx}$. The previous monotonicity property does not hold for f_q^∞, as the continuous quasiconvex functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = \frac{|x|}{1+|x|}$ and $f_2(x) \equiv 1$ show.

An analytic formula for the qx-asymptotic function is given below.

Proposition 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc and quasiconvex function, then for any $u \in \mathbb{R}^n$ we have*

$$f^{qx}(u) = \inf_{x \in \mathbb{R}^n} \sup_{t \ge 0} f(x + tu). \tag{13}$$

Proof For all $\lambda > f^{qx}(u)$ we have $u \in (S_\lambda(f))^\infty$. Then we can find $x \in S_\lambda(f)$, such that for all $t \ge 0$ we have $x + tu \in S_\lambda(f)$. Thus, there exists $x \in \mathbb{R}^n$ such that $\sup_{t\ge0} f(x + tu) \le \lambda$, which implies that $\inf_{x\in\mathbb{R}^n} \sup_{t\ge0} f(x + tu) \le \lambda$. This is true for all $\lambda > f^{qx}(u)$. Thus, $\inf_{x\in\mathbb{R}^n} \sup_{t\ge0} f(x + tu) \le f^{qx}(u)$.

Conversely, if $\inf_{x\in\mathbb{R}^n}\sup_{t\geq 0} f(x+tu) < \lambda$, then there exists $x \in \mathbb{R}^n$ such that for all $t \geq 0$, $x + tu \in S_\lambda(f)$. Hence $u \in (S_\lambda(f))^\infty$, so $f^{qx}(u) \leq \lambda$.

This shows that $f^{qx}(u) \leq \inf_{x\in\mathbb{R}^n}\sup_{t\geq 0} f(x+tu)$ and proves equality (13). □

Remark 3.2 Let $C \subseteq \mathbb{R}^n$ be a closed and convex set. Then $(\delta_C)^{qx} = \delta_{C^\infty}$. For the usual asymptotic function, a similar result is [1, Corollary 2.5.1].

Another analytic formula for f^{qx} is given below.

Proposition 3.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc and quasiconvex function. Then for each $u \in \mathbb{R}^n$,*

$$f^{qx}(u) = \inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} f(x+tu). \tag{14}$$

Proof We know by [13] that for a quasiconvex function defined on an interval I in $\mathbb{R}$, there exist two consecutive disjoint intervals I_1, I_2 (one of them might be empty) with $I = I_1 \cup I_2$, such that the function is decreasing on I_1 and increasing on I_2. Thus,

$$\sup_{t\geq 0} f(x+tu) = \max\left\{f(x), \lim_{t\to+\infty} f(x+tu)\right\}.$$

Since obviously

$$f^{qx}(u) = \inf_{x\in\mathbb{R}^n} \max\left\{f(x), \lim_{t\to+\infty} f(x+tu)\right\} \geq \inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} f(x+tu), \tag{15}$$

in order to show (14) it is enough to show that strict inequality in (15) is not possible. Assume that strict inequality holds. Then there exists $x_0 \in \mathbb{R}^n$ such that $\lim_{t\to+\infty} f(x_0 + tu) < f^{qx}(u)$.

Take t_0 large enough so that $f(x_0 + t_0 u) < f^{qx}(u)$. Set $x_1 := x_0 + t_0 u$. Then obviously $\lim_{t\to+\infty} f(x_1 + tu) = \lim_{t\to+\infty} f(x_0 + tu)$. Thus,

$$\max\left\{f(x_1), \lim_{t\to+\infty} f(x_1 + tu)\right\} < f^{qx}(u) = \inf_{x\in\mathbb{R}^n} \max\left\{f(x), \lim_{t\to+\infty} f(x + tu)\right\},$$

which is a contradiction. □

Remark 3.3 From (13) we get that $f^{qx}(0) = \inf_{x\in\mathbb{R}^n} f(x)$. Also from the same formula, for all $u \in \mathbb{R}^n$ we have $f^{qx}(u) \geq \inf_{x\in\mathbb{R}^n} f(x)$. Consequently,

$$\inf_{x\in\mathbb{R}^n} f(x) = \inf_{u\in\mathbb{R}^n} f^{qx}(u) = f^{qx}(0). \tag{16}$$

Hence, f and f^{qx} has the same optimal value, the qx-asymptotic function obtains the optimal value at $u = 0$.

From the geometric point of view, the qx-asymptotic function provides the behavior of the value of the original quasiconvex function at infinity, rather than the behavior of the slope, as f^∞ does. The next example ilustrates our interpretation.

Example 3.1 Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous quasiconvex function given by

$$f(x) = \begin{cases} x^2, & x \leq 0, \\ \frac{x}{1+x}, & x > 0. \end{cases}$$

An easy calculation shows that

$$f^{qx}(u) = \begin{cases} +\infty, & u < 0, \\ 0, & u = 0, \\ 1, & u > 0. \end{cases}$$

The following proposition provides calculus rules for the qx-asymptotic function. We recall that the composition of an increasing function h with a quasiconvex function g is also quasiconvex.

Proposition 3.3 *The following assertions hold,*

(a) *Let* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lsc and quasiconvex function, and* $h : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *be an increasing continuous function such that* $\operatorname{dom} h \cap g(\mathbb{R}^n) \neq \emptyset$. *We extend* h *to* $\overline{\mathbb{R}}$ *by setting* $h(\pm\infty) = \lim_{t\to\pm\infty} h(t)$. *Then* $(h \circ g)^{qx} = h(g^{qx})$.

(b) *Let* $f_i : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ *be a family of proper, lsc and quasiconvex functions with* I *an arbitrary index set. Then*

$$\left(\sup_{i\in I} f_i\right)^{qx} \geq \sup_{i\in I} (f_i)^{qx}. \tag{17}$$

Proof (a) : Obviously, $h \circ g$ is proper, lsc and quasiconvex. Take $u \in \mathbb{R}^n$, then

$$\begin{aligned}(h \circ g)^{qx}(u) &= \inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} (h \circ g)(x + tu) = \inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} h(g(x + tu)) \\ &= \inf_{x\in\mathbb{R}^n} h\left(\lim_{t\to+\infty} g(x + tu)\right) = h\left(\inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} g(x + tu)\right) \\ &= h(g^{qx}(u)).\end{aligned}$$

(b) Set $f := \sup_{i\in I} f_i$. Then

$$f^{qx}(u) = \inf_{x\in\mathbb{R}^n} \sup_{t\geq 0} \sup_{i\in I} f_i(x+tu) = \inf_{x\in\mathbb{R}^n} \sup_{i\in I} \sup_{t\geq 0} f_i(x+tu)$$

$$\geq \sup_{i\in I} \inf_{x\in\mathbb{R}^n} \sup_{t\geq 0} f_i(x+tu) = \sup_{i\in I} (f_i)^{qx}(u).$$

Hence (17) holds. □

Note that in general equality does not hold in (17).

Example 3.2 Define on $\mathbb{R}^2$ the convex functions given by $f_1(x_1,x_2) = |x_1 - 1|$ and $f_2(x_1,x_2) = |x_1+1|$, and $f = \max\{f_1, f_2\} = 1 + |x_1|$. Take $u = (0,1)$. Then

$$(f_1)^{qx}(u) = \inf_{(x_1,x_2)\in\mathbb{R}^2} \sup_{t\geq 0} |x_1 - 1| = 0, \ \ (f_2)^{qx}(u) = \inf_{(x_1,x_2)\in\mathbb{R}^2} \sup_{t\geq 0} |x_1 + 1| = 0,$$

$$f^{qx}(u) = \inf_{(x_1,x_2)\in\mathbb{R}^2} \sup_{t\geq 0} (1 + |x_1|) = 1.$$

Thus, $f^{qx}(u) > \max\{(f_1)^{qx}(u), (f_2)^{qx}(u)\}$.

Another formula for computing the qx-asymptotic function is given below.

Proposition 3.4 *Let $g : \mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ be a proper, lsc and quasiconvex function, let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with $A(\mathbb{R}^n)\cap \operatorname{dom} g \neq \emptyset$, and let $f(x) := g(Ax)$. Then f is lsc, quasiconvex and*

$$f^{qx}(u) \geq g^{qx}(Au), \ \forall\, u \in \mathbb{R}^n. \tag{18}$$

Whenever A is onto, equality holds in (18).

Proof It is clear that f is lsc and quasiconvex. Now, take any $u \in \mathbb{R}^n$, then

$$\begin{aligned} f^{qx}(u) &= \inf_{x\in\mathbb{R}^n}\sup_{t\geq 0} f(x+tu) = \inf_{x\in\mathbb{R}^n}\sup_{t\geq 0} g(Ax+t(Au)) \\ &\geq \inf_{z\in\mathbb{R}^m}\sup_{t\geq 0} g(z+t(Au)) = g^{qx}(Au). \end{aligned}$$

If A is onto, then Ax takes on all values $z \in \mathbb{R}^m$ so equality holds. □

3.2 Comparison with Other Asymptotic Functions

Let us compare the three asymptotic functions f^∞, f_q^∞ and $f^\infty(\cdot;\lambda)$ that are known from the literature, with the function f^{qx} introduced in the previous subsection.

When f is convex, the three functions f^∞, f_q^∞ and $f^\infty(\cdot;\lambda)$ are equal, see also [9, Proposition 5.4]:

Proposition 3.5 *Let f be convex and λ be such that $S_\lambda(f) \neq \emptyset$. Then*

$$f^\infty = f_q^\infty = f^\infty(\cdot;\lambda).$$

Proof Only $f^\infty = f^\infty(\cdot;\lambda)$ needs a proof. We note that for $x \in S_\lambda(f)$, the functions $t \mapsto \frac{f(x+tu)-f(x)}{t}$ and $t \mapsto \frac{f(x)-\lambda}{t}$ are increasing, thus $t \mapsto \frac{f(x+tu)-\lambda}{t}$ is increasing too, and

$$\sup_{t>0}\frac{f(x+tu)-\lambda}{t} = \lim_{t\to+\infty}\frac{f(x+tu)-\lambda}{t} = \lim_{t\to+\infty}\frac{f(x+tu)-f(x)}{t} = f^\infty(u).$$

It follows that

$$f^{\infty}(u;\lambda) = \sup_{x\in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu)-\lambda}{t} = f^{\infty}(u),$$

and the proof is complete. $\square$

In contrast to the above, when f is convex, f^{qx} is in general not equal to f^{∞}. For example, consider the constant function $f(x) := \alpha$. Here, $f^{\infty} \equiv 0$ and $f^{qx} \equiv \alpha$. Hence, for $\alpha > 0$ we have $f^{\infty} < f^{qx}$, while for $\alpha < 0$ we have $f^{qx} < f^{\infty}$. The same example shows that there is no connection between f^{qx} and f^{∞}_q or $f^{\infty}(\cdot;\lambda)$. This difference is not surprising, since f^{∞} is related to the slope of the function f at infinity, whereas f^{qx} is related to the value of f at infinity.

The qx-asymptotic function f^{qx} is also convex whenever f is convex. In fact, it is constant in its domain:

Proposition 3.6 *Let f be proper, convex and lsc. Then $f^{qx} = \inf f + \delta_C$, where $C := \{u \in \mathbb{R}^n : f^{\infty}(u) \leq 0\}$.*

Proof By [1, Proposition 2.5.3], for each $\alpha \in \mathbb{R}$ such that $S_\alpha(f) \neq \emptyset$, one has the equality: $(S_\alpha(f))^{\infty} = S_0(f^{\infty})$. Thus, f^{qx} has just one sublevel set, and its value on this sublevel set is $\inf f^{qx} = \inf f$. $\square$

The asymptotic functions f^{qx}, f^{∞} and $f^{\infty}(\cdot;\lambda)$ are not convex in general if f is not convex. In contrast, f^{∞}_q is always convex, for any proper function f. To see this, we first recall the notion of recession cone of an arbitrary set [15,16].

Definition 3.2 Let K be any nonempty subset of $\mathbb{R}^n$. Its recession cone is the set

$$\operatorname{rec} K := \{u \in \mathbb{R}^n : \ x + tu \in K, \ \forall\, x \in K, \ \forall\, t > 0\}.$$

Note that K is not required to be closed or convex. If K is closed and convex, then $\operatorname{rec} K = K^\infty$, the usual asymptotic cone of K.

It is known (see [15, Exercise 6.34], or [16, Lemma 2.1]) that for any nonempty set K from $\mathbb{R}^n$, the set $\operatorname{rec} K$ is always a convex cone.

A natural definition is the following.

Definition 3.3 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. We define the (generalized) recession function of f as the function $f^{rec} : \mathbb{R}^n \to \overline{\mathbb{R}}$ for which

$$\operatorname{epi} f^{rec} := \operatorname{rec}(\operatorname{epi} f). \tag{19}$$

The recession function is well-defined, as shown by the following proposition, which is useful to understand the nature of the q-asymptotic function.

Proposition 3.7 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then for every $u \in \mathbb{R}^n$*

$$f^{rec}(u) = \sup_{x \in \operatorname{dom} f} \sup_{t>0} \frac{f(x+tu) - f(x)}{t} = f^\infty_q(u). \tag{20}$$

Proof Observe that

$$
\begin{aligned}
(u,\alpha) \in \operatorname{rec}(\operatorname{epi} f) &\Longleftrightarrow (x,\lambda) + t(u,\alpha) \in \operatorname{epi} f,\ \forall\ (x,\lambda) \in \operatorname{epi} f,\ \forall\ t > 0 \\
&\Longleftrightarrow (x+tu, f(x)+t\alpha) \in \operatorname{epi} f,\ \forall\ x \in \operatorname{dom} f,\ \forall\ t > 0 \\
&\Longleftrightarrow f(x+tu) \le f(x) + t\alpha,\ \forall\ x \in \operatorname{dom} f,\ \forall\ t > 0 \\
&\Longleftrightarrow \frac{f(x+tu)-f(x)}{t} \le \alpha,\ \forall\ x \in \operatorname{dom} f,\ \forall\ t > 0 \\
&\Longleftrightarrow \sup_{x\in \operatorname{dom} f} \sup_{t>0} \frac{f(x+tu)-f(x)}{t} \le \alpha \\
&\Longleftrightarrow (u,\alpha) \in \operatorname{epi} f_q^{\infty}.
\end{aligned}
$$

This shows that $\operatorname{rec}(\operatorname{epi} f) = \operatorname{epi} f_q^{\infty}$, so f^{rec} is well defined and is equal to f_q^{∞}. □

As a result, we have:

Proposition 3.8 *For any proper function f, its q-asymptotic function f_q^{∞} is convex.*

Proof Set $K := \operatorname{epi} f$, by [16, Lemma 2.1] or [15, Exercise 6.34] we have that $\operatorname{rec}(\operatorname{epi} f) = \operatorname{epi} f_q^{\infty}$ is convex. Thus, f_q^{∞} is convex. □

Remark 3.4 The λ-asymptotic function $g := f^{\infty}(\cdot;\lambda)$ satisfies

$$S_0(g) = \operatorname{rec}(S_\lambda(f)).$$

Indeed, $u \in \mathrm{rec}(S_\lambda(f))$ is equivalent to $x + tu \in S_\lambda(f)$ for all $x \in S_\lambda(f)$ and all $t > 0$. This is equivalent to $f(x + tu) \le \lambda,\ \forall\, x \in S_\lambda(f),\ \forall\, t > 0$, that is,

$$\sup_{x \in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu) - \lambda}{t} \le 0.$$

This in turn means $u \in S_0(g)$.

As seen in Proposition 3.6, the qx-asymptotic function f^{qx} is very particular when the function f is proper, convex and lsc. However, in some situations, even in this case, f^{qx} gives us information about the behavior of the function at infinity while other asymptotic functions fail to do so.

Example 3.3 Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be the proper, lsc and convex function given by $f(x) = -\sqrt{x}$ for $x \ge 0$, and $f(x) = +\infty$ otherwise. Here

$$f^\infty(u) = f_q^\infty(u) = f^\infty(u; \lambda) = 0,\ u \ge 0,$$

and no information about the unboundedness from below of f was detected. On the other hand, for $u > 0$ we have $f^{qx}(u) = -\infty$. Which means that f is not bounded from below.

4 Applications in Optimization

In this section, applications for quasiconvex optimization problems are given. We analyze the link between our new results with previous ones for the convex case. We also show that our new asymptotic function has some properties that previous quasiconvex asymptotic functions do not have.

The next proposition is straightforward, since f and f^{qx} have the same infimum and f^{qx} attains its infimum at 0.

Proposition 4.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Then f is bounded from below iff $f^{qx} > -\infty$.*

A characterization result for boundedness from below for convex functions using first and second order asymptotic functions can be found in [10, Section 3.3].

The qx-asymptotic function characterizes the boundedness of a quasiconvex function as the next proposition shows. For the convex case, a related result is [1, Proposition 2.5.5].

Proposition 4.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Then f is bounded iff f^{qx} is real-valued.*

Proof If f is bounded, then obviously f^{qx} is real-valued, by formula (14). Conversely, assume that f^{qx} is real-valued, then f is bounded from below by the previous proposition. For showing that f is bounded from above, we observe that since f^{qx} is real-valued, $\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} S_k(f^{qx})$.

As the sets $S_k(f^{qx})$ are closed, by Baire's theorem there exists $k_0 \in \mathbb{N}$ such that the interior of $S_{k_0}(f^{qx})$ is nonempty. Thus, there exist $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq S_{k_0}(f^{qx})$. Now, let $m \in \mathbb{N}$ be such that $m > \max\{f^{qx}(-x_0), k_0\}$. Then $-x_0 \in S_m(f^{qx})$ and $B(x_0, \varepsilon) \subseteq S_m(f^{qx})$, thus $\operatorname{conv}(\{-x_0\} \cup B(x_0, \varepsilon)) \subseteq S_m(f^{qx})$. It follows that $0 \in \operatorname{int} S_m(f^{qx})$ and since $S_m(f^{qx})$ is a cone, $S_m(f^{qx}) = \mathbb{R}^n$. Since $S_m(f^{qx}) = (S_m(f))^\infty$, then $S_m(f) = \mathbb{R}^n$, so f is bounded from above by m and the result follows. □

Remark 4.1 We notice that the previous proposition does not hold for the q-asymptotic function. In fact, set $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \min\{\sqrt{|x|}, 3\}$, which is continuous, bounded and quasiconvex. Here $f_q^\infty(u) = +\infty$ for all $u \neq 0$. On the other hand, for the function $f(x) = |x|$, the function f_q^∞ is real valued, but f is unbounded.

The next result provides a characterization of the nonemptiness and compactness of the solution set of a lsc quasiconvex function.

Theorem 4.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Then the following assertions are equivalent.*

(a) $\operatorname{argmin}_{\mathbb{R}^n} f$ *is nonempty and compact.*

(b) $\operatorname{argmin}_{\mathbb{R}^n} f^{qx}$ *is nonempty and compact.*

(c) $f^{qx}(u) > f^{qx}(0)$ *for all* $u \neq 0$.

Proof Obviously (c) implies (b). If (b) holds and $u_0 \in \operatorname{argmin}_{\mathbb{R}^n} f^{qx}$, then $tu_0 \in \operatorname{argmin}_{\mathbb{R}^n} f^{qx}$ for all $t > 0$ since f^{qx} is 0-homogeneous. Hence necessarily $u_0 = 0$, so (c) holds.

$(c) \Rightarrow (a)$: If (a) does not hold, then there exists a sequence (x_k) with $f(x_k) \to \inf_{\mathbb{R}^n} f$ and $\|x_k\| \to +\infty$. By selecting a subsequence if necessary, we may assume that $\frac{x_k}{\|x_k\|} \to u$. For every $\lambda > \inf_{\mathbb{R}^n} f$ we have that $x_k \in S_\lambda(f)$ for k large enough, so $u \in (S_\lambda(f))^\infty = S_\lambda(f^{qx})$, that is, $f^{qx}(u) \leq \lambda$. Hence, $f^{qx}(u) \leq \inf_{\mathbb{R}^n} f = f^{qx}(0)$, contradicting (c).

$(a) \Rightarrow (c)$: Suppose for the contrary that (c) does not hold. It follows that there exists $u \neq 0$ such that $f^{qx}(u) \leq f^{qx}(0) = \inf_{\mathbb{R}^n} f$. Then $u \in S_{\inf f}(f^{qx}) = (S_{\inf f}(f))^{\infty}$. Choose $x \in \operatorname{argmin}_{\mathbb{R}^n} f$. Then $x \in S_{\inf f}(f)$. Thus for every $t > 0$, we have that $x + tu \in S_{\inf f}(f)$. This implies that $x + tu \in \operatorname{argmin}_{\mathbb{R}^n} f$, which contradicts the compactness of $\operatorname{argmin}_{\mathbb{R}^n} f$. $\square$

Remark 4.2 Since for a proper, lsc and convex function $f^{\infty}(0) = 0$, the previous characterization for quasiconvexity is similar to the characterization (5). Another similar characterization for quasiconvexity is (8) (see [9, Theorem 4.7]) since $f_q^{\infty}(0) = 0$.

In the next example, we study the quasiconvex quadratic case, that is the case when the function f is given by $f(x) := \frac{1}{2}\langle x, Ax\rangle + \langle a, x\rangle + \alpha$, where $A \in \mathbb{R}^{n\times n}$ is symmetric, a belongs to $\mathbb{R}^n$ and α belongs to $\mathbb{R}$. To that end, we first recall that whenever f is quadratic, f is convex on $\mathbb{R}^n$ iff f is quasiconvex on $\mathbb{R}^n$ (see [12, Theorem 6.3.1]). Thus, a quadratic function f can be quasiconvex without being convex, only if its domain is a proper subset K of $\mathbb{R}^n$. We say that f is *merely quasiconvex* on K if f is a quasiconvex function without being convex on K [12, page 120]. If $\operatorname{int} K$ is nonempty, a necessary condition for a quadratic f to be merely quasiconvex is the existence of exactly one simple negative eigenvalue of A (see [12, Remark 6.3.1]). The properties of quasiconvex quadratic functions are investigated in depth in [12, Chapter 6].

Example 4.1 Let K be a nonempty closed, convex and proper subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ a quasiconvex quadratic function $f(x) = \frac{1}{2}\langle x, Ax\rangle + \langle a, x\rangle + \alpha$.

As usual, we extend f to the whole of $\mathbb{R}^n$ by setting $f(x) = +\infty$ for $x \notin K$. Observe that if $x \in K$ and $u \in K^\infty$, then

$$f(x+tu) = f(x) + t\langle \nabla f(x), u\rangle + \frac{1}{2}t^2\langle u, Au\rangle.$$

Accordingly, by Proposition 3.2,

$$f^{qx}(u) = \inf_{x\in K} \lim_{t\to+\infty} \left(f(x) + t\left\langle \nabla f(x), u\right\rangle + \frac{1}{2}t^2\left\langle u, Au\right\rangle \right).$$

- If $\langle u, Au\rangle > 0$, then the limit equals $+\infty$ for all $x \in K$, so $f^{qx}(u) = +\infty$.
- If $\langle u, Au\rangle < 0$, then $f^{qx}(u) = -\infty$, so $\inf f = -\infty$.
- If $\langle u, Au\rangle = 0$, then the limit is $+\infty$ for all $x \in K$ such that $\langle \nabla f(x), u\rangle > 0$. These x can be omitted from the calculation of the infimum. In case $\langle \nabla f(x), u\rangle < 0$ for some $x \in K$, then $f^{qx}(u) = -\infty$.

Thus, f^{qx} is given by the following formula:

$$f^{qx}(u) = \begin{cases} +\infty, & \text{if } \langle u, Au\rangle > 0, \\ -\infty, & \text{if } \langle u, Au\rangle < 0, \\ -\infty, & \text{if } \langle u, Au\rangle = 0 \text{ and } u \notin \left\{\nabla f(K)\right\}^*, \\ \inf\limits_{x\in K,\ \langle \nabla f(x), u\rangle = 0} f(x), & \text{if } \langle u, Au\rangle = 0 \text{ and } u \in \left\{\nabla f(K)\right\}^*. \end{cases}$$

Remark 4.3

(i) Characterizations for the nonemptiness and compactness of the solution set for quasiconvex quadratic functions are well-known. See for instance [17, Theorem 4.6] where the authors use the q-asymptotic function.

(ii) The term $\langle u, Au\rangle$ is exactly the second order asymptotic function $f^{\infty\infty}$ introduced in [17] (see [17, Example 3.5]).

We provide another classical example for a class of nonconvex functions.

Example 4.2 Consider the affine functions defined by $h(x) = \langle a, x\rangle + \alpha$ and $g(x) = \langle b, x\rangle + \beta$ with $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, and the closed and convex set $K := \{x \in \mathbb{R}^n : \ g(x) \geq 1\}$. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the linear fractional function

$$f(x) = \begin{cases} \dfrac{h(x)}{g(x)}, \text{ if } x \in K, \\ +\infty, \ \text{if } x \notin K. \end{cases}$$

It is well-known that f is semistrictly quasiconvex on K and quasiconvex (see [12,18]), and $K^{\infty} = \{u \in \mathbb{R}^n : \ \langle b, u\rangle \geq 0\}$. Notice that, for $u \in K^{\infty}$,

$$f^{qx}(u) = \inf_{x \in K} \lim_{t \to +\infty} f(x + tu) = \inf_{x \in K} \lim_{t \to +\infty} \frac{h(x) + t\langle a, u\rangle}{g(x) + t\langle b, u\rangle}. \tag{21}$$

We have three cases:

(i) If $\langle b, u\rangle > 0$, then it is easy to see that $f^{qx}(u) = \frac{\langle a,u\rangle}{\langle b,u\rangle}$.

(ii) If $\langle b, u\rangle < 0$, then for t sufficiently large, $x + tu \notin K$ so $f(x + tu) = +\infty$. In this case, $f^{qx}(u) = +\infty$.

(iii) If $\langle b, u\rangle = 0$, then again we have three cases: For $\langle a, u\rangle > 0$ we find from (21) that $f^{qx}(u) = +\infty$. For $\langle a, u\rangle < 0$ we find $f^{qx}(u) = -\infty$. Finally, for $\langle a, u\rangle = 0$, relation (21) gives

$$f^{qx}(u) = \inf_{x \in K} \frac{h(x)}{g(x)} = \inf_{x \in K} f(x) = f^{qx}(0).$$

Before we introduce our next proposition, we remind that for a proper lsc convex function, $f^\infty(0) = 0$ so $f^\infty(u) \leq 0$ is equivalent to $f^\infty(u) \leq f^\infty(0)$. Also, for proper lsc quasiconvex functions, one has $f^{qx}(0) = \inf f = \inf f^{qx}$ so $f^{qx}(u) \leq f^{qx}(0)$ is equivalent to $f^{qx}(u) = f^{qx}(0)$.

Proposition 4.3 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function and $u \in \mathbb{R}^n$. Then $f^{qx}(u) = f^{qx}(0)$ iff for every $x \in \mathrm{dom} f$, the function $t \mapsto f(x + tu),\ t > 0$ is decreasing.*

Proof For $u = 0$ it is obvious, so we assume that $u \neq 0$.

($\Rightarrow$) Take $x \in \mathrm{dom} f$. Since $f^{qx}(u) = f^{qx}(0) = \inf f$, we have $f^{qx}(u) \leq f(x)$ so $u \in S_{f(x)}(f^{qx}) = \big(S_{f(x)}(f)\big)^\infty$. From $x \in S_{f(x)}(f)$, for every $t \geq 0$ we have that $x + tu \in S_{f(x)}(f)$, that is, $f(x + tu) \leq f(x)$. Thus, for every $x \in \mathrm{dom} f$ and every $t > 0$ we have $f(x+tu) \leq f(x)$. For every $t' > t > 0$, set $x' := x+tu$ and $t'' := t' - t$. Then we have $f(x' + t''u) \leq f(x')$, so $f(x + t'u) \leq f(x + tu)$. Consequently, the function $t \mapsto f(x + tu),\ t > 0$ is decreasing.

($\Leftarrow$) Assume that for each $x \in \mathrm{dom} f$, the function $t \mapsto f(x + tu),\ t > 0$ is decreasing. Suppose to the contrary that $f^{qx}(u) > f^{qx}(0)$. As $f^{qx}(0) = \inf f$, we can choose $x \in \mathrm{dom} f$ with $f(x) < f^{qx}(u)$. Then

$$f(x) < f^{qx}(u) \leq \lim_{t \to +\infty} f(x + tu).$$

It follows that $t \mapsto f(x+tu),\ t > 0$ cannot be decreasing, a contradiction. $\square$

Remark 4.4 If f is a proper, lsc and convex function, then the following assertions are equivalent: $f^\infty(u) \leq 0$; for each $x \in \mathrm{dom} f$, the function

$t \mapsto f(x+tu),\ t>0$ is decreasing (see [1, Theorem 2.5.2] and [19, Theorem 8.6]); for some (equivalently, for every) $x \in \mathrm{dom} f$, $\lim_{t \mapsto +\infty} f(x+tu) < +\infty$.

Note that for a convex function, or more generally for a quasiconvex function, $t \mapsto f(x+tu),\ t>0$ is monotone for large values of t, so the limit always exist and the lim inf or lim sup used in [1, Theorem 2.5.2] and [19, Theorem 8.6] are not needed. In contrast with convex functions, if f is quasiconvex, it is possible that $t \mapsto f(x+tu),\ t>0$ is decreasing only for some $x \in \mathrm{dom} f$. For example, consider the quasiconvex function $f(x) = \min\{\|x\| - 1, 0\}$, $x \in \mathbb{R}^2$. If e_1 and e_2 are the usual basis vectors, then $t \mapsto f(e_2 + te_1),\ t>0$ is decreasing, while $t \mapsto f(-e_1 + te_1),\ t>0$ is not.

Now, we will recall [10, Theorem 3.1]. To this end, we first recall the following class of functions. As a consequence of [19, Theorem 8.6], this class includes those functions that are convex or coercive.

Definition 4.1 ([10, Definition 3.1]) A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be in $\mathcal{C}$ if for all $x \in \mathrm{dom} f$ and $u \in (\mathrm{dom} f)^\infty$, the function $t \mapsto f(x+tu)$, $t>0$, is either unbounded from above or decreasing.

Now the mentioned theorem.

Theorem 4.2 ([10, Theorem 3.1]) *Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a vector function with each f_j, $j = 1, 2, \dots, m$, being a continuous, semistrictly quasiconvex function belonging to $\mathcal{C}$, and let $K \subseteq \mathbb{R}^n$ be closed and convex. Assume that*

$$L_j := \{u \in K^\infty : \ (f_j)_q^\infty(u) \leq 0\}, \tag{22}$$

is a linear subspace for all $j \in \{1, 2, \dots, m\}$*. Then* $F(K)$ *is closed.*

Using the qx-asymptotic function, we can rewrite the previous theorem. First, we express $\mathcal{C}$ in terms of f^{qx}.

Proposition 4.4 *A proper, lsc and quasiconvex function belongs to* $\mathcal{C}$ *iff for every* $u \in \mathbb{R}^n$, $f^{qx}(u) = +\infty$ *or* $f^{qx}(u) = f^{qx}(0)$.

Proof If for every $u \in \mathbb{R}^n$ one has $f^{qx}(u) = +\infty$ or $f^{qx}(u) = f^{qx}(0)$, then by Propositions 3.1 and 4.3, the function $t \mapsto f(x + tu)$ is either unbounded or decreasing for all $x \in \operatorname{dom} f$.

Conversely, assume that for each $x \in \operatorname{dom} f$ and $u \in (\operatorname{dom} f)^\infty$, the function $t \mapsto f(x + tu)$ is either unbounded from above or decreasing. Then also for each $u \in \mathbb{R}^n$ the function $t \mapsto f(x + tu)$ is either unbounded from above or decreasing, because for $u \notin (\operatorname{dom} f)^\infty$, one has $x + tu \notin \operatorname{dom} f$ for t large. In fact, if this is not true, then there is a sequence $t_n \to +\infty$ such that $x_n := x + t_n u \in \operatorname{dom} f$. But then $\lim \frac{(x_n - x)}{t_n} = \lim \frac{x_n}{t_n} = u \in (\operatorname{dom} f)^\infty$, a contradiction.

Now, assume that $f^{qx}(0) < f^{qx}(u) < +\infty$. Since $\inf f = f^{qx}(0)$, we may choose x such that $f(x) < f^{qx}(u)$. From Proposition 3.2 it follows that $\lim_{t \to +\infty} f(x + tu) \geq f^{qx}(u) > f(x)$, so the function $t \mapsto f(x + tu)$ is not decreasing. Also, for $\lambda > f^{qx}(u)$, Definition 3.1 shows that $u \in (S_\lambda(f))^\infty$ so $x + tu \in S_\lambda(f)$ for all $t > 0$. This means that $t \mapsto f(x + tu)$ is neither decreasing, nor unbounded from above, a contradiction. □

From the definition of f_q^∞ and Proposition 4.3, it is clear that for a proper, lsc and quasiconvex function f, $f_q^\infty(u) \leq 0$ is equivalent to $f^{qx}(u) = f^{qx}(0)$. Using then Proposition 4.4, we can rewrite Theorem 4.2 as follows:

Corollary 4.1 *Let $F := (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a vector function with each f_j, $j = 1, 2, \ldots, m$, being a continuous and semistrictly quasiconvex function. Let $K \subseteq \mathbb{R}^n$ be a closed and convex set. Assume that for every $j \in \{1, 2, \ldots, m\}$, one has $(f_j)^{qx}(u) = +\infty$ or $(f_j)^{qx}(u) = (f_j)^{qx}(0)$ for all $u \in K^\infty$, and*

$$(L_j)^{qx} := \{u \in K^\infty : \ (f_j)^{qx}(u) = (f_j)^{qx}(0)\}, \tag{23}$$

is a linear subspace for all $j \in \{1, 2, \ldots, m\}$. Then $F(K)$ is a closed set.

Notice that the same result was written only in terms of the qx-asymptotic function and no class of functions was used.

5 Conclusions

The qx-asymptotic function has been proved to be useful to deal with the family of quasiconvex functions. This function preserves important properties and calculus rules, which were ensured only convex functions when the usual asymptotic function is used. The applications show that the description of the value of a quasiconvex function at infinity is enough for the study of the boundedness of the function in the whole effective domain, for the characterization of the nonemptiness and compactness of the set of minimizers, and also for providing elegant sufficient conditions for closedness criteria.

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