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A Characterization by Optimization of the MONGE Point of a Tetrahedron

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Abstract *"... nihil omnino in mundo contingint, in quo non maximi minimive ratio quapiam eluceat"*, translated into *"... nothing in all the world will occur in which no maximum or minimum rule is somehow shining forth",* used to say L.EULER in 1744. This is confirmed by numerous applications of mathematics in physics, mechanics, economy, etc. In this note, we show that it is also the case for the classical "centres" of a tetrahedron, more specifically for the so-called MONGE point (the substitute of the notion of orthocentre for a tetrahedron). To the best of our knowledge, the characterization of the MONGE point of a tetrahedron by optimization, that we are going to present, is new.

Keywords Tetrahedron · Monge point · quadratic convex function · variational principle

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1 Introduction: To Begin with... What Kind of Tetrahedron?

Let $\mathscr{T} = ABCD$ be a tetrahedron in the three dimensional space $\mathbb{R}^3$(equipped with the usual Euclidean and affine structures); the points A, B, C, D are supposed not to lie in a plane, of course. We begin with two particular types of tetrahedra and, then, with increase in generality, we can classify the tetrahedra into several classes. Here they are:

- The *regular tetrahedron*. This tetrahedron enjoys so many symmetries that it is not very interesting from the optimization viewpoint: all the “centres” usually associated with a tetrahedron (and that we are going to visit again in the next paragraph) coincide.

- The *trirectangular tetrahedra*. They are generalizations to the space of rectangular triangles in the plane. A trirectangular tetrahedron $OABC$ has (two by two) three perpendicular faces OBC, OAB, OAC and a “hypothenuse-face” ABC; such a tetrahedron enjoys a remarkable relationship between areas of its faces (see [1]); its vertex O, opposite the hypothenuse-face, is the orthocentre and MONGE point, as we shall see below.

- The *orthocentric tetrahedra*. Curiously enough, the four altitudes of a tetrahedron generally do not meet at a point; when this happens, the tetrahedron is called orthocentric. A common characterization of orthocentric tetrahedra is as follows: a tetrahedron is orthocentric if and only if the opposite edges (two by two) are orthogonal. This class of tetrahedra is by far the most studied one in the literature. Regular and trirectangular tetrahedra are indeed orthocentric.

- *General tetrahedra*. Like for triangles, three specific “centres” can be defined for any tetrahedron: the centroid or isobarycentre, the incentre and the circumcentre. We shall see their characterization by optimization, as for some other points, in the next section. As said before, the altitudes do not necessarily meet at a point; moreover, the projection of any vertex on the opposite face does not necessarily coincide with the orthocentre of this face. The notion of orthocentre will be held by a new point: the so-called MONGE point.

2 Characterization by Optimization of the Centroid, Incentre, Circumcentre, etc. of a Tetrahedron

Let us revisit the usual centres of a tetrahedron $\mathscr{T} = ABCD$ and their characterizations by optimization. They are similar to the ones developed for the centres of a triangle in [2].

- The *centroid* or *isobarycentre* (also called *centre of mass*, *centre of gravity*) G of $\mathscr{T}$ is the best known of them: it can be defined as the barycentre of the four vertices A, B, C, D with equal weights. In terms of optimization, it is the point which minimizes (on $\mathscr{T}$ or on $\mathbb{R}^3$) the following objective function or criterion

$$P \longmapsto f_1(P) = (PA)^2 + (PB)^2 + (PC)^2 + (PD)^2. \tag{1}$$

Should f_1 represent a temperature function, the point G would be the coolest point. Indeed, f_1 is a quadratic and strictly convex function. According to the well-known necessary and sufficient condition for optimality in unconstrained convex minimization, the G point is characterized by the vectorial relation $\overrightarrow{\nabla f_1}(G) = \overrightarrow{0}$, which can be written here as

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} = \overrightarrow{0}. \tag{2}$$

- The *incentre* I of $\mathscr{T}$ is the centre of the largest sphere included in $\mathscr{T}$; it is also the point (in $\mathscr{T}$) equidistant from the four faces of $\mathscr{T}$. In terms of optimization, it is the point which minimizes (on $\mathscr{T}$ or on $\mathbb{R}^3$) the following function

$$P \longmapsto f_2(P) = \max(PA', PB', PC', PD'), \tag{3}$$

where A', B', C', D' denote the projections of P on the faces of $\mathscr{T}$ (A' lies in the face opposite the vertex A, and so on). This new temperature function f_2 is again strictly convex, but nondifferentiable.

- The *circumcentre* O is the centre of the smallest sphere containing $\mathscr{T}$; it is also the point equidistant from the four vertices of $\mathscr{T}$. When it lies in the interior of $\mathscr{T}$, this point is the one which minimizes on $\mathscr{T}$ (as also on $\mathbb{R}^3$) the following function

$$P \longmapsto f_3(P) = \max(PA, PB, PC, PD). \tag{4}$$

As for f_2, this function f_3 is strictly convex and nondifferentiable.

The above characterizations of the incentre and circumcentre in terms of optimization do not seem to be well-known. The difficulty there is that they involve a nonsmooth convex function (to be minimized), while the function to be minimized in (1) is convex and smooth. However, people involved in approximation theory are familiar with such kinds of minimization problems. We add three further interesting points to our list, even far less known however.

- The first LEMOINE *point* of $\mathscr{T}$ is the one which minimizes on $\mathscr{T}$ the function

$$P \longmapsto f_4(P) = (PA')^2 + (PB')^2 + (PC')^2 + (PD')^2, \tag{5}$$

where A', B', C', D' denote, as above, the projections of P on the faces of $\mathscr{T}$. See [3, page 79] for another characterization of this point.

- The second LEMOINE *point* of $\mathscr{T}$ is, this time, the one which minimizes on $\mathscr{T}$ the function

$$P \longmapsto f_5(P) = (P\,P_{AB})^2 + (P\,P_{AC})^2 + (P\,P_{AD})^2 + (P\,P_{BC})^2 + (P\,P_{BD})^2 + (P\,P_{CD})^2, \tag{6}$$

where P_{AB}, P_{AC},... denote the projections of P on the six edges of $\mathscr{T}$.

In the case of a triangle, there is only one LEMOINE point ([3, page 24]); for a tetrahedron, there are two.

- The FERMAT or FERMAT & TORRICCELLI *point* of $\mathscr{T}$ is the one which minimizes the function

$$P \longmapsto f_6(P) = PA + PB + PC + PD. \tag{7}$$

It is usually located inside the tetrahedron, but it could also be one of the vertices.

Again, the functions appearing in (5), (6) and (7) are convex, but the one in (7) is not differentiable at the points A, B, C or D. So, characterizing the FERMAT point is different if a vertex of the tetrahedron is a candidate for optimality or not; however, subdifferential calculus and optimality conditions for convex functions, be they smooth or nonsmooth, cover all the cases [4, chapter D].

3 Properties of the Centroid: an Example of Reasoning with Convexity

Here we illustrate how techniques from convex analysis allow us to derive easily some properties of “centres” in a tetrahedron; for this, we choose the particular case of the centroid. Let $x_A, \ldots, x_D$ denote the vectors of cartesian coordinates of the points $A,...,D$ and x_G that of the barycentre G of $(A,\alpha),(B,\beta),(C,\gamma),(D,\delta)$. Here, the weights $\alpha,...,\delta$ are those of a convex combination (*i.e.,* positive and summing up to 1); for the centroid, these weights are exactly equal. We express x_G as a convex combination of $x_A,...,x_D$:

$$x_G = \alpha x_A + \beta x_B + \gamma x_C + \delta x_D. \tag{8}$$

We now proceed like in the proofs of results in convex analysis (page 28 in [4] for example) and transform (8) into:

$$x_G = (\alpha+\beta)(\frac{\alpha}{\alpha+\beta}x_A + \frac{\beta}{\alpha+\beta}x_B) + (\gamma+\delta)(\frac{\gamma}{\gamma+\delta}x_C + \frac{\delta}{\gamma+\delta}x_D); \tag{9}$$

or

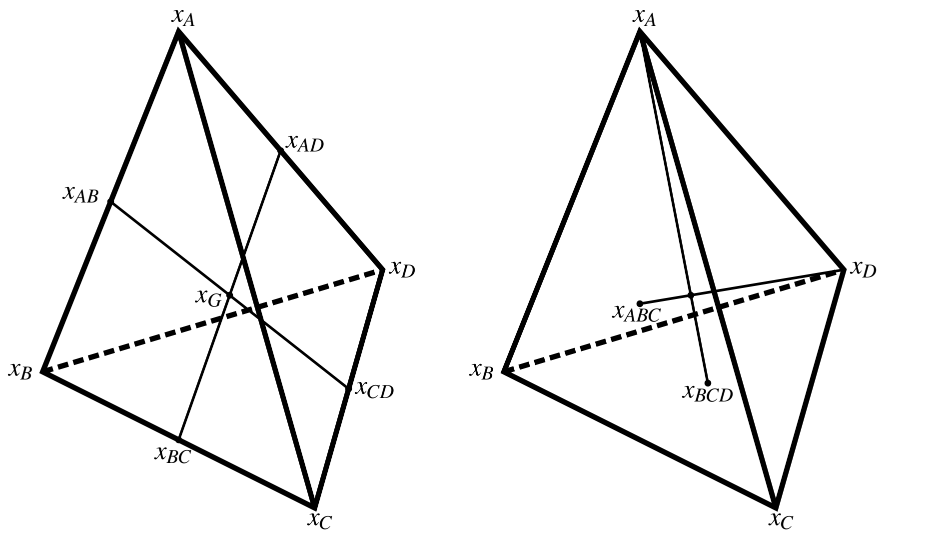
$$x_G = (\alpha+\beta+\gamma)(\frac{\alpha}{\alpha+\beta+\gamma}x_A + \frac{\beta}{\alpha+\beta+\gamma}x_B + \frac{\gamma}{\alpha+\beta+\gamma}x_C) + \delta x_D. \tag{10}$$

In (9): $\frac{\alpha}{\alpha+\beta}x_A + \frac{\beta}{\alpha+\beta}x_B =: x_{AB}$ lies on the line-segment AB (it is the midpoint of AB in the case of centroid), $\frac{\gamma}{\gamma+\delta}x_C + \frac{\delta}{\gamma+\delta}x_D =: x_{CD}$ lies on the line-segment CD; therefore, the x_G appears as a new convex combination of x_{AB} and x_{CD} (it is again the mid-point in the case of centroid). In (10) the point $\frac{\alpha}{\alpha+\beta+\gamma}x_A + \frac{\beta}{\alpha+\beta+\gamma}x_B + \frac{\gamma}{\alpha+\beta+\gamma}x_C =: x_{ABC}$ lies on the triangle ABC (it is its centroid in the case where $\alpha=\beta=\gamma=\delta=1/4$). In doing so, we have proved, without referring to any geometric argument (angles, distances, vector calculus) that:

- the centroid x_G lies exactly in the middle of the line-segments joining midpoints of edges (there are three situations like this);

- the centroid x_G is situated on the lines joining vertices to centroids of the opposite triangles (there are four situations like this), the corresponding barycentric weights being $3/4$ and $1/4$.

See Figure 1 for a graphical illustration.



Joining midpoints of edges   Joining vertices to centroids of triangles

Fig. 1

4 The Monge Point

By an extraordinary geometrical intuition, G.MONGE (1746-1818) proposed a substitute for the notion of orthocentre of a tetrahedron. Here is its basic definition. *Consider the six planes perpendicular to the edges of a tetrahedron $\mathscr{T}$ and passing through the midpoints of the respective opposite sides; then these six planes meet at just one point; the point common to these planes is the so-called* MONGE *point of the tetrahedron.* Our question is: *can the* MONGE *point be viewed as the minimizer on $\mathscr{T}$ of some appropriate criterion ?* In other words, what is the temperature function on $\mathscr{T}$ (a convex, possibly differentiable, function) such that the coolest point is exactly the MONGE point? Before answering this question, we review the main properties of the MONGE point; they can be found in some classical books on solid geometry, or in the paper [5] that we recommend.

- If the tetrahedron is orthocentric, the orthocentre does exist as the intersection of the four altitudes, and the MONGE point coincides with it. So, the MONGE point is the direct generalization of the notion of orthocentre to arbitrary tetrahedra.

- If G denotes the centroid of the tetrahedron and O its circumcentre, the MONGE point M satisfies the following vectorial property: $\overrightarrow{OM} = 2\overrightarrow{OG}$. In other words, the MONGE point is the symmetric of the circumcentre with respect to the centroid.

- Another geometrical construction. A.MANNHEIM's theorem (1895): the four planes determined by the four altitudes of a tetrahedron and the orthocentres of the corresponding faces pass through the MONGE point. For example, in a trirectangular tetrahedron $OABC$ (see section 1), the MONGE point is the vertex O.

- The MONGE point, more precisely its projection on faces, holds a compromise between the projections of vertices on faces and orthocentres of faces: the MONGE point is equidistant from the orthocentre of a face and the projection on it of the opposite vertex (that holds true for the four faces). Again, in the case of orthocentric tetrahedra, the orthocentre of a face and the projection on it of the opposite vertex coincide; this common point is the projection of the orthocentre-MONGE point of the tetrahedron.

Now, we make the definition of MONGE point "variational", that is to say in terms of optimization. Our procedure will illustrate the following well-known adage: *"This problem, when solved, will be simple"*. So, for each edge σ, denote by $\overrightarrow{v_\sigma}$ a *unitary* vector directing σ (for example $(x_B - x_A)/\,\|x_B - x_A\|$ for the edge joining the vertices A and B) and x_σ the midpoint of the opposite edge ; then, since our aim is to look for a point x such that $\langle x - x_\sigma, \overrightarrow{v_\sigma}\rangle = 0$ for all σ, define the function

$$P \text{ (or vector } x) \longmapsto f_7(P) = \frac{1}{2}\sum_{\sigma_1}^{\sigma_6} \left[\langle x - x_\sigma, \overrightarrow{v_\sigma}\rangle\right]^2, \tag{11}$$

Theorem 4.1 *The f_7 function is quadratic and strictly convex. It is uniquely minimized at the* MONGE *point.*

Proof As a sum of squares of affine forms, the f_7 function is clearly quadratic. The gradient vector $\nabla f_7(x)$ and the hessian matrix $\nabla^2 f_7(x)$ of f_7 at x are as follows:

$$\nabla f_7(x) = \sum_{\sigma_1}^{\sigma_6} \langle x - x_\sigma, \overrightarrow{v_\sigma}\rangle \; \overrightarrow{v_\sigma}; \tag{12}$$

$$\nabla^2 f_7(x) = \sum_{\sigma_1}^{\sigma_6} \overrightarrow{v_\sigma}.(\overrightarrow{v_\sigma})^T.$$

To show that $\nabla^2 f_7(x)$ is positive definite, consider the quadratic form

$$h \in \mathbb{R}^3 \longmapsto q(h) = \langle \nabla^2 f_7(x)h, h\rangle = \sum_{\sigma_1}^{\sigma_6} \langle \overrightarrow{v_\sigma}, h\rangle^2.$$

Clearly, $q(h)$ is nonnegative on $\mathbb{R}^3$ and $q(h) = 0$ if and only if $h = 0$ (that is due to the fact that three among the six vectors $\overrightarrow{v_\sigma}$ are linearly independent). Hence the f_7 function is strictly convex.

The MONGE point is indeed the unique minimizer of the nonnegative function f_7 since it is the only point where f_7 achieves the 0 value. □

Remark 4.1 As the proof clearly shows, it suffices to consider in (11) a summation over *three* linearly independent edges, for example the three edges arising from a vertex.

Remark 4.2 The f_7 function contains distances (to points or lines) in a hidden form. Indeed, consider the following construction: for each edge σ, let Δ_σ be the line passing through (the opposite midpoint) x_σ and parallel to σ; we call it the *mirror (edge or) line* to σ. Then, because the vectors $\overrightarrow{v_\sigma}$ have been chosen unitary, the square of the distance from x to the mirror line Δ_σ is

$$d^2(x, \Delta_\sigma) = \|x - x_\sigma\|^2 - [\langle x - x_\sigma, \overrightarrow{v_\sigma}\rangle]^2 . \tag{13}$$

So, the f_7 function is

$$P \text{ (or vector } x) \longmapsto f_7(P) = \frac{1}{2}\sum_{\sigma_1}^{\sigma_6}\left[\|x - x_\sigma\|^2 - d^2(x, \Delta_\sigma)\right] . \tag{14}$$

Other substitutes for the criterion to be minimized could be the following functions:

$$P \text{ (or vector } x) \longmapsto f_8(P) = \sum_{\sigma_1}^{\sigma_6}[\|x - x_\sigma\| - d(x, \Delta_\sigma)] ; \tag{15}$$

$$P \text{ (or vector } x) \longmapsto f_9(P) = \max_{\sigma_1,\dots,\sigma_6} \ [\|x - x_\sigma\| - d(x, \Delta_\sigma)] . \tag{16}$$

Hence, minimizing the f_8 function or the f_9 function yields again the MONGE point. However, in comparison with the f_7 function, we loose properties like convexity and differentiability (on $\mathbb{R}^3$).

Remark 4.3 What about the numerical computation of the Monge point ? We may use the (primary) definitions of this point as the intersection of appropriate planes. The results dealing with optimization, presented above, are also easily amenable to computations: either to solve the system $\nabla f_7(x) = 0$ or, better, to minimize f_7 (summation over only three independent edges). In that case, the stopping rule of any minimization algorithm is: $|f_7(x_k)| \leqslant \varepsilon$. In all cases, simple computations with MATLAB work well.

Remark 4.4 Back to the triangle ([6]). Analogous functions to those presented above can be defined to find (by minimization) the orthocentre of an acute triangle ABC. If σ is one side of the triangle, x_σ the vertex opposite to it, $\overrightarrow{v_\sigma}$ a unitary vector directing σ, and Δ_σ the line passing through x_σ and parallel to σ, then the line orthogonal to σ and passing through x_σ has for equation: $\langle x - x_\sigma, \overrightarrow{v_\sigma} \rangle = 0$, or $\|x - x_\sigma\| = d(x, \Delta_\sigma)$. As a result, the three functions f_7, f_8, f_9 (where the summations or the max operation are taken over the threes sides of the triangle) are minimized on $\mathbb{R}^2$ at a unique point: the orthocentre of the triangle. Actually, the f_8 function coincides on $\mathscr{T}$ with the one presented in [6], minus a constant (which equals the sum of the three altitudes of the triangle). This means, at least for an acute triangle: in order to find the orthocentre of a triangle, instead of minimizing the sum of distances from a point P to the vertices, plus the sum of distances from P to the sides, one could minimize the difference of two sums: the sum of the distances from P to the vertices, and the sum of the distances from P to the mirror sides (*i.e.*, parallels to the sides, which are drawn through the vertices). That amounts to minimizing

$$\begin{aligned} P \text{ (or vector } x) \longmapsto f_{10}(P) &= PA + PB + PC - (d_1 + d_2 + d_3) \\ &= (PA + PB + PC) + (PH_1 + PH_2 + PH_3) - (e_1 + e_2 + e_3), \end{aligned}$$

where the e_1, e_2, e_3 stand for the distances between the three sides and their mirrors (or the lengths of the three altitudes). See Figure 2.

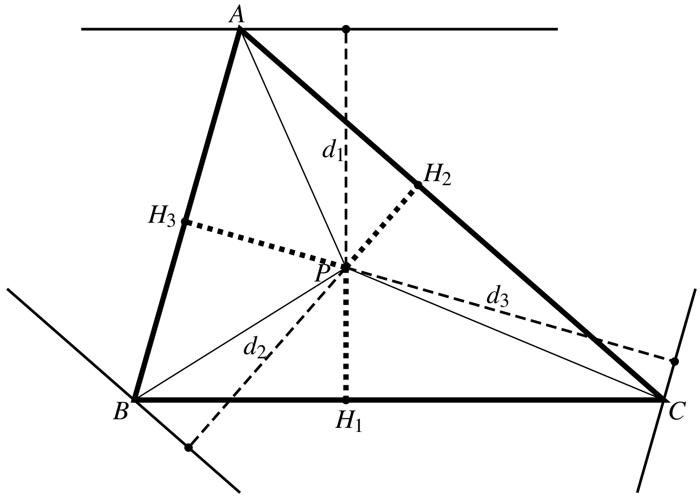


Fig. 2. The case of a triangle

5 Generalization to n Dimensions

Consider $n+1$ points in $\mathbb{R}^n$ that generate a convex polyhedron of full dimension n. The $\binom{n+1}{n}$ hyperplanes that pass through the barycentres of any $n-1$ points and are orthogonal to the edge passing through the remaining two, meet exactly at one point. This point is called again the MONGE point of the polyhedron; it has been studied recently in [7, 8]. This point can also be found by minimizing a strictly convex function like in (11).

6 Conclusion

Among the usual centres of a tetrahedron, the Monge point has a particular flavor. In our main theorem (Theorem 4.1), we have proved that the Monge point of a tetrahedron is the unique minimizer of a quadratic strictly convex function, which is expressed in terms of distances to vertices and edges.

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