Translations of Quasimonotone Maps and Monotonicity

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Abstract– We show that given a convex subset K of a topological vector space X and a multivalued map $T : K \rightrightarrows X^*$, if there exists a straight line S which is not perpendicular to K and such that $T + w$ is quasimonotone for each $w \in K$, then T is monotone. No differentiability or even continuity assumption is imposed, thus generalizing some recent results in the literature.

Keywords– Quasimonotone map, pseudomonotone map, monotone map.

Let X be a real topological vector space, X^* be its dual space, and $K \subseteq X$ be nonempty and convex. A multivalued map $T : K \rightrightarrows X^*$ is called pseudomonotone (in the Karamardian's sense) [1] if for every $x, y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$, the following implication holds:

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0$$

while it is called quasimonotone if the following implication holds:

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

A monotone map is pseudomonotone, while a pseudomonotone map is quasimonotone. The converse is not true.

If T is pseudomonotone (resp., quasimonotone) and $w \in X^*$, then $T + w$ is not pseudomonotone (resp., quasimonotone) in general. In case of a single-valued, linear map T defined on the whole space $\mathbb{R}^n$, it is known for instance that if $T + w$ is quasimonotone, then T is monotone [2]. Other results in this direction are given in [3].

Very recently, He [4] and Isac and Motreanu [5] studied the more general case of a nonlinear map T and showed the following interesting result: Let X be a Hilbert space ($X = \mathbb{R}^n$ for [4]) and K be a convex subset of X with nonempty interior. If $T : K \to X$ is a single-valued, continuous map which is Gâteaux differentiable on the interior of K and such that $T + w$ is quasimonotone for all w in a straight line S, then T is monotone. In both papers, differentiability is essential since the argument is based on first-order characterizations of generalized monotonicity found in [6, 7].

The purpose of this paper is to extend this result to multivalued maps defined on a convex subset of a real topological vector space. No assumption of differentiability or even continuity will be made, and the domain need not have a nonempty interior, thus permitting the application of the result to domains such as the positive cone of l^p or L^p, $p \geq 1$.

Before stating the main result, we recall some definitions. Given $v \in X^*$ and a nonempty set $K \subseteq X$, we say that v is perpendicular to K if v is constant on K, i.e., $\langle v, x\rangle = \langle v, y\rangle$ for all $x, y \in K$. If $S = \{u + tv : t \in \mathbb{R}\}$, $u \in X^*$,

$v \in X^*\backslash\{0\}$ is a straight line in X^*, we say that S is perpendicular to K if v is perpendicular to K. Given $x, y \in X$, we denote by $[x, y]$ the line segment $\{(1-t)\,x + ty : t \in [0,1]\}$.

Theorem 1 *Let X be a real topological vector space, $K \subseteq X$ be nonempty and convex and $T : K \rightrightarrows X^*$ be a map. Assume that there exists a straight line $S = \{u + tv : t \in \mathbb{R}\}$ in X^*, not perpendicular to K, which is such that for every $w \in S$, $T + w$ is quasimonotone. Then T is monotone.*

We first prove a lemma:

Lemma 2 *Assumptions as in Theorem 1. Then the restriction of T on any line segment of K which is not perpendicular to S is monotone.*

Proof. Let l be a line segment of K which is not perpendicular to S. Assume that there exist $x, y \in l$, $x^* \in T(x)$ and $y^* \in T(y)$ such that $\langle y^* - x^*, y - x\rangle < 0$. This means that

$$\langle y^*, y - x\rangle < \langle x^*, y - x\rangle .$$

Note that $\langle v, y - x\rangle \neq 0$, otherwise v would be constant on l. It follows that the range of the function $g(t) := -\langle u, y - x\rangle - t\,\langle v, y - x\rangle$ is equal to $\mathbb{R}$; hence there exists $t_0 \in \mathbb{R}$ such that

$$\langle y^*, y - x\rangle < -\langle u, y - x\rangle - t_0\,\langle v, y - x\rangle < \langle x^*, y - x\rangle .$$

Setting $w = u + t_0 v$, we deduce that $\langle x^* + w, y - x\rangle > 0$ while $\langle y^* + w, y - x\rangle < 0$, thus contradicting the quasimonotonicity of the map $T + w$. ■

Proof. (of Theorem 1). Let $x, y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$ be arbitrary. If $\langle v, y - x\rangle \neq 0$, then S is not perpendicular to $[x, y]$, thus by the lemma T is monotone on $[x, y]$ and

$$\langle y^* - x^*, y - x\rangle \geq 0. \tag{1}$$

Now assume that $\langle v, y - x\rangle = 0$. Since S is not perpendicular to K, we may choose some $z \in K$ such that $\langle v, z\rangle \neq \langle v, x\rangle = \langle v, y\rangle$. For all $s \in (0,1)$ set $z_s = sz + (1-s)\,\frac{x+y}{2}$. Then S is not perpendicular to the line segments $[x, z_s]$, $[y, z_s]$ and $\left[\frac{x+y}{2}, z\right]$, thus its restriction to each of these line segments is monotone. We deduce that for any $z_s^* \in T(z_s)$ and $z^* \in T(z)$ the following inequalities hold:

$$\langle z_s^*, z_s - x\rangle \geq \langle x^*, z_s - x\rangle \tag{2}$$

$$\langle z_s^*, z_s - y\rangle \geq \langle y^*, z_s - y\rangle \tag{3}$$

$$\langle z^*, z - z_s\rangle \geq \langle z_s^*, z - z_s\rangle . \tag{4}$$

From (2) and (3) it follows that

$$\langle z_s^*, 2z_s - (x + y)\rangle \geq \langle x^*, z_s - x\rangle + \langle y^*, z_s - y\rangle$$

thus

$$2s\left\langle z_s^*, z - \frac{x+y}{2}\right\rangle \geq \langle x^*, z_s - x\rangle + \langle y^*, z_s - y\rangle . \tag{5}$$

Since $z - z_s = (1-s)\left(z - \frac{x+y}{2}\right)$, (4) implies

$$\left\langle z^*, z - \frac{x+y}{2} \right\rangle \geq \left\langle z_s^*, z - \frac{x+y}{2} \right\rangle. \tag{6}$$

Combining (5) with (6) we obtain

$$2s \left\langle z^*, z - \frac{x+y}{2} \right\rangle \geq \langle x^*, z_s - x \rangle + \langle y^*, z_s - y \rangle.$$

Letting $s \to 0$ and taking into account that $z_s \to \frac{x+y}{2}$ we obtain again (1), i.e., T is monotone. ■

We note that the assumption "S is not perpendicular to K" is automatically satisfied if K has nonempty interior. We recall that $x_0 \in K$ is called a quasi-interior point [8] if there exists no supporting hyperplane of K at x_0, i.e., if there exists no $x^* \in X^* \backslash \{0\}$ such that for all $x \in K$, $\langle x^*, x - x_0 \rangle \geq 0$. It is clear that if there exists some $v \in X^* \backslash \{0\}$ which is perpendicular to K, then K has no quasi-interior points. Hence the assumption "S is not perpendicular to K" is also automatically satisfied if K has nonempty quasi-interior. For instance the positive cone l_+^p for $p \geq 1$ has empty interior, but its quasi-interior is nonempty.

Finally we note that the assumption "S is not perpendicular to K" cannot be omitted from the theorem. Indeed, set $X = \mathbb{R}^2$, $K = \mathbb{R} \times \{0\}$, and $S = \{0\} \times \mathbb{R}$. If we define $T : K \to \mathbb{R}^2$ by $T(x, 0) = e^{-x}(1, 0)$, then T is not monotone, while $T + w$ is pseudomonotone for every $w \in S$. Hence, even if we assume in Theorem 1 that $T + w$ is pseudomonotone (rather than quasimonotone) for all $w \in S$, the assumption of S not being perpendicular to K is still necessary.

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