Set-Valued and Variational Analysis

Some conditions for maximal monotonicity of bifunctions

--Manuscript Draft--

Manuscript Number:	SVAA-D-15-00007R1
Full Title:	Some conditions for maximal monotonicity of bifunctions
Article Type:	Original research
Keywords:	Monotone bifunctions; maximal monotone operators; equilibrium problems
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Funding Information:	King Fahd University of Petroleum and Minerals (SA) (SR141001) — Prof. Nicolas Hadjisavvas
	Conselho Nacional de Desenvolvimento Científico e Tecnológico (BR) (246699/2012-4) — Flávia Morgana O. Jacinto
	Ministerio de Ciencia e Innovación (ES) (MTM2011-29064-C03-01) — Juan Enrique Martínez-Legaz
	Australian Research Council (AU) (DP140103213) — Juan Enrique Martínez-Legaz
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Set-Valued and Variational Analysis manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract We present necessary and sufficient conditions for a monotone bifunction to be maximally monotone, based on a recent characterization of maximally monotone operators. These conditions state the existence of solutions to equilibrium problems obtained by perturbing the defining bifunction in a suitable way.

Mathematics Subject Classification (2010) 47H05 · 47N10

1 Introduction

This paper deals with monotonicity in the context of a reflexive Banach space X. Monotonicity for set valued operators and bifunctions appear in the literature as an elegant and

The first author was supported by the startup research grant no. SR141001 of the King Fahd University of Petroleum and Minerals. This work was developed during the postdoctoral stay of the second author at the Department of Economics and Economic History of the Universitat Autònoma de Barcelona. She gratefully acknowledges the financial support received from Science without Borders Program of CNPq-Brazil under grant 246699/2012-4 and the warm hospitality of the third author as well as of the members from the host institution. The third author was supported by the MICINN of Spain, Grant MTM2011-29064-C03-01, and under Australian Research Council's Discovery Projects funding scheme (project number DP140103213).

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useful mathematical notion. Monotone bifunctions have been extensively studied since the following equilibrium problem appeared in the seminal work of Blum and Oettli [3]: given a bifunction $F : X \times X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ and a set $K \subseteq X$ such that $F(x,x) = 0$ for every $x \in K$, find $\bar{x} \in K$ such that $F(\bar{x}, y) \geqq 0$ for all $y \in K$.

The aim of this paper is to present some necessary and sufficient conditions for a monotone bifunction to be maximal in the pointwise sense. The conditions we will present are equivalent to the existence of solutions to a class of equilibrium problems obtained by perturbing a given bifunction in a suitable way.

The paper is organized as follows. In the next section, we begin fixing notations and reviewing some important points of monotone operator theory. Notation not explicitly defined there is standard and as in [11]. Based on the definitions in [1] and [5], we introduce suitable notions of monotone and maximally monotone bifunctions and show that there is a bijection between special classes of monotone bifunctions and monotone operators. We finish the section by introducing pointwise maximal monotonicity. In Section 3, we use a recent characterization of maximally monotone operators due to the third author [9], in order to characterize maximally monotone bifunctions in terms of the solution sets of equilibrium problems. We also get a generalization of the following characterization of maximality: a monotone operator A is maximally monotone if and only if for each $x \in X$, there exists $x' \in X$ such that $0 \in J(x' - x) + A(x')$, with J denoting the duality mapping (see, for instance, [6, p. 324]). Our last result extends [5, Proposition 2.6].

2 Notation and Preliminary Results

In the following, X is a reflexive Banach space, X^* its dual and $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ is the duality pairing. The *indicator function* of a set $C \subseteq X$ is the function $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$$

Given a multivalued operator $T : X \rightrightarrows X^*$, its *domain* and *graph* are, respectivelly, the sets

$$D(T) = \{x \in X : T(x) \neq \emptyset\}$$
$$\mathrm{Graph}(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}.$$

The operator is called *monotone* if for every $x, y \in X$ and $x^* \in T(x)$, $y^* \in T(y)$,

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

It is called *maximally monotone* if it is monotone and its graph is not properly included in the graph of any other monotone operator. If the inequality is strict whenever $x \neq y$, we say that T is *strictly monotone.* The *inverse* of T is the operator $T^{-1} : X^* \rightrightarrows X$ defined by

$$T^{-1}(x^*) = \{x \in X : x^* \in T(x)\}$$

Given a proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, its *Fenchel conjugate* is the function $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}.$$

The *subdifferential* of a l.s.c. proper convex function f at a point $x \in X$ is the set

$$\partial f(x) = \{x^* \in X^* : f(y) \geq f(x) + \langle y - x, x^* \rangle, \quad \forall y \in X\}.$$

This defines an operator $\partial f : X \rightrightarrows X^*$ which is maximally monotone [12]. If we take f to be the function $\frac{1}{2}\|\cdot\|^2$, then its subdifferential operator is a maximally monotone operator called the *duality mapping* of X; it is denoted by J, and satisfies

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}; \tag{1}$$

thus, $J(-x) = -J(x)$ for all $x \in X$.

For any operator $T : X \rightrightarrows X^*$ with $D(T) \neq \emptyset$, the Fitzpatrick function of T is the function $\varphi_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi_T(x, x^*) = \langle x, x^* \rangle - \inf_{(y,y^*) \in \mathrm{Graph}(T)} \langle x - y, x^* - y^* \rangle .$$

If T is maximally monotone, then $\varphi_T \geq \langle \cdot, \cdot \rangle$ with equality exactly on $\mathrm{Graph}\, T$; moreover, it is the smallest convex function having this property.

Given $(x_0, x_0^*) \in X \times X^*$, the translation $\tau_{(x_0,x_0^*)}(T)$ of T is the operator defined by

$$\tau_{(x_0,x_0^*)}(T)(x) = T(x + x_0) - x_0^*.$$

Monotonicity and maximal monotonicity is preserved by the translation. In addition,

$$\varphi_{\tau_{(x_0,x_0^*)}(T)}(x, x^*) = \varphi_T(x + x_0, x^* + x_0^*) - [\langle x, x_0^* \rangle + \langle x_0, x^* \rangle + \langle x_0, x_0^* \rangle] \tag{2}$$

(see Section 3.1 in [10]).

We will make use of the following result from [9]:

Theorem 1 *For every monotone operator $T : X \rightrightarrows X^*$, the following statements are equivalent:*

(a) T is maximally monotone;

(b) $\mathrm{Graph}(T) + \mathrm{Graph}(-B) = X \times X^*$ *for every maximally monotone operator $B : X \rightrightarrows X^*$ such that φ_B is finite-valued.*

(c) There exists a maximally monotone operator $B : X \rightrightarrows X^*$ such that φ_B is finite-valued and $\mathrm{Graph}(T) + \mathrm{Graph}(-B) = X \times X^*$, *and there exists* $(p, p^*) \in \mathrm{Graph}(B)$ *such that* $\langle p - y, p^* - y^* \rangle > 0$ *for every* $(y, y^*) \in \mathrm{Graph}(B) \backslash \{(p, p^*)\}$.

Remark 1 As it is obvious from the proof of the above theorem in [9], the assumptions that B is monotone (and in particular maximally monotone) and φ_B is finite-valued, are not used in the proof of the implication (c) $\Rightarrow$ (a). Hence, if there exists an operator $B : X \rightrightarrows X^*$ such that $\mathrm{Graph}(T) + \mathrm{Graph}(-B) = X \times X^*$ and there exists $(p, p^*) \in \mathrm{Graph}(B)$ such that $\langle p - y, p^* - y^* \rangle > 0$ for every $(y, y^*) \in \mathrm{Graph}(B) \backslash \{(p, p^*)\}$, then T is maximally monotone.

A bifunction is, by definition, any function $F : X \times X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$. The domain of a bifunction F is defined to be the set

$$\mathrm{dom}\, F = \{x \in X : \forall y \in X,\ F(x, y) > -\infty\}.$$

Definition 1 A bifunction $F : X \times X \to \overline{\mathbb{R}}$ is called monotone if

$$F(x, y) \leq -F(y, x),\ \forall x, y \in X.$$

If F is monotone, then for every $x,y \in \operatorname{dom} F$ one will have

$$-\infty < F(x,y) \le -F(y,x) < +\infty$$

so $F(x,y), F(y,x) \in \mathbb{R}$ and $F(x,y)+F(y,x) \le 0$. Further, for every $x \in X$, $F(x,x) \le -F(x,x)$ implies that $F(x,x) \le 0$ since $F(x,x) = +\infty$ is impossible.

Given any bifunction $F : X \times X \to \overline{\mathbb{R}}$ one defines the operator $A^F : X \rightrightarrows X^*$ by

$$A^F(x) = \{x^* \in X^* : \langle y-x, x^* \rangle \le F(x,y),\ \forall y \in X\}.$$

It is obvious that $D(A^F) \subseteq \operatorname{dom} F$. As it is clear from its definition, if $F(x,x) = 0$ then $A^F(x) = \partial F(x,\cdot)(x)$ (the Fenchel subdifferential of the function $F(x,\cdot)$ at the point x). This happens in particular whenever $x \in D(A^F)$ and F is monotone; indeed, in this case for any $x^* \in A^F(x)$ we have $F(x,x) \ge \langle x-x, x^* \rangle = 0$. Since $F(x,x) \le 0$ by monotonicity, we deduce $F(x,x) = 0$.

A^F corresponds to one of the two "diagonal subdifferential operators" introduced in [7], see also [8].

Remark 2 It is easy to check that whenever a bifunction F is monotone, the operator A^F is also monotone. The converse is not true, as can be seen by the following example. Define $F : X \times X \to \mathbb{R}$ by $F(x,y) = \|y-x\|^2$. Then F is not monotone, and it is easy to show that $A^F = 0$ so it is monotone. This leads to other examples, by taking the sum of the above bifunction with another monotone bifunction. For instance, if $T : X \to X^*$ is any monotone, bounded linear operator, then the bifunction $G(x,y) = \langle Tx, y-x \rangle + \|T\|\,\|y-x\|^2$ is not monotone and $A^G = T$.

For every operator $T : X \rightrightarrows X^*$ one defines the bifunction G_T by

$$G_T(x,y) = \sup_{x^* \in T(x)} \langle y-x, x^* \rangle .$$

Obviously, $G_T(x,y) = -\infty$ if and only if $T(x) = \emptyset$. Thus, $D(T) = \operatorname{dom} G_T$.

The mappings $T \mapsto G_T$ and $F \mapsto A^F$ are not one-to-one [5]; however, if we restrict their domain and range, they become bijections as we shall see. We introduce first the following notation:

- $\mathscr{C}(X^*) := \{C^* \subseteq X^* : C^* \text{ is a closed convex set}\}$
- $\mathscr{O}(X) := \{T : X \rightrightarrows X^* : T(x) \in \mathscr{C}(X^*),\ \forall x \in X\}$
- $\mathscr{O}_m(X) := \{T \in \mathscr{O}(X) : T \text{ is monotone}\}$
- $\mathscr{S}(X) := \{s : X \to \overline{\mathbb{R}} : s \equiv -\infty, \text{ or } s(0) = 0 \text{ and } s \text{ is l.s.c., convex and positively homogenous}\}$
- $\mathscr{B}_s(X) := \{F : X \times X \to \overline{\mathbb{R}} : F(x, x+\cdot) \in \mathscr{S}(X),\ \forall x \in X\}$
- $\mathscr{B}_m(X) := \{F \in \mathscr{B}_s(X) : F \text{ is monotone}\}$.

Note that for every $s \in \mathscr{S}(X)$, either $s \equiv -\infty$ or $s(x) > -\infty$ for all $x \in X$.

If T is a maximally monotone operator, then $T \in \mathscr{O}_m(X)$. Also, if F is a monotone bifunction, then $A^F \in \mathscr{O}_m(X)$.

For every $C^* \in \mathscr{C}(X^*)$, recall that its support function is

$$\sigma_{C^*} := \delta^*_{C^*}.$$

It is easy to check that $\sigma_{C^*} \in \mathscr{S}(X)$. The mapping $\mathscr{C}(X^*) \ni C^* \mapsto \sigma_{C^*} \in \mathscr{S}(X)$ so defined is a bijection; its inverse is the mapping $\psi : \mathscr{S}(X) \to \mathscr{C}(X^*)$ defined by

$$\psi(s) = \{x^* : \langle x, x^* \rangle \leq s(x), \quad \forall x \in X\}.$$

This means that

$$\psi(\delta^*_{C^*}) = C^* \text{ and } \delta^*_{\psi(s)} = s, \forall C^* \in \mathscr{C}(X^*), \quad \forall s \in \mathscr{S}(X). \tag{3}$$

This bijection satisfies $\psi(-\infty) = \emptyset$.

For every $T \in \mathscr{O}(X)$ one has

$$G_T(x, x+\cdot) = \sup_{x^* \in T(x)} \langle \cdot, x^* \rangle = \delta^*_{T(x)} \tag{4}$$

so $G_T \in \mathscr{B}_s(X)$. Conversely, for every $F \in \mathscr{B}_s(X)$, it is clear that $A^F \in \mathscr{O}(X)$. Note that

$$A^F(x) = \{x^* \in X^* : \langle d, x^* \rangle \leq F(x, x+d), \, \forall d \in X\} = \psi\left(F(x, x+\cdot)\right). \tag{5}$$

The following simple proposition holds:

Proposition 1 *For every $T \in \mathscr{O}(X)$, we have $A^{G_T} = T$. Also, for every $F \in \mathscr{B}_s(X)$, we have $G_{A^F} = F$. Consequently, the mapping*

$$\mathscr{O}(X) \ni T \mapsto G_T \in \mathscr{B}_s(X)$$

is a bijection, whose inverse is the mapping

$$\mathscr{B}_s(X) \ni F \mapsto A^F \in \mathscr{O}(X).$$

The restriction of this mapping to $\mathscr{O}_m(X)$ is a bijection between $\mathscr{O}_m(X)$ and $\mathscr{B}_m(X)$.

Proof For every $T \in \mathscr{O}(X)$ and $x \in X$ we have, using successively (5), (4) and (3):

$$A^{G_T}(x) = \psi(G_T(x, x+\cdot)) = \psi(\delta^*_{T(x)}) = T(x).$$

Similarly, for every $F \in \mathscr{B}_s(X)$ and $x \in X$ we obtain:

$$G_{A^F}(x, x+\cdot) = \delta^*_{A^F(x)} = \delta^*_{\psi(F(x, x+\cdot))} = F(x, x+\cdot).$$

Thus, $G_{A^F} = F$. The remaining assertions follow immediately. □

A usual definition of maximal monotonicity for bifunctions is the following:

Definition 2 A monotone bifunction $F : X \times X \to \overline{\mathbb{R}}$ is said to be a maximally monotone bifunction (MMB) if the operator A^F is maximally monotone.

We also introduce another definition, on a more restricted class of bifunctions:

Definition 3 A bifunction $F \in \mathscr{B}_m(X)$ is said to be a pointwise maximally monotone bifunction (PMMB) if F is pointwise maximal in $\mathscr{B}_m(X)$, that is,

$$\forall H \in \mathscr{B}_m(X), \quad F \leq H \Rightarrow F = H$$

The following proposition shows the relation between the two notions:

Proposition 2 *Let $F \in \mathscr{B}_m(X)$. Then F is an MMB if and only if F is a PMMB.*

Proof Let $F \in \mathscr{B}_m(X)$ be a PMMB, and let T be a maximally monotone extension of A^F. Then $A^F(x) \subseteq T(x)$ for all $x \in X$, hence $G_{A^F} \leq G_T$. By Proposition 1, $F = G_{A^F}$, thus $F \leq G_T$. Since F is a PMMB and $G_T \in \mathscr{B}_m(X)$, we obtain $F = G_T$. Hence, $A^F = A^{G_T} = T$ by Proposition 1. Hence A^F is maximally monotone and F is an MMB.

Conversely, let $F \in \mathscr{B}_m(X)$ be an MMB. Assume that $H \in \mathscr{B}_m(X)$ satisfies $F \leq H$. Then $A^F(x) \subseteq A^H(x)$ for all $x \in X$. Since A^F is a maximally monotone operator, $A^F = A^H$. In view of Proposition 1, this implies that

$$F = G_{A^F} = G_{A^H} = H.$$

Hence, F is a PMMB. $\square$

Given any bifunction F, its Fitzpatrick transform [1,2] is defined as the function $\Phi_F : X \times X^* \to \overline{\mathbb{R}}$ given by

$$\Phi_F(x,x^*) = (-F(\cdot,x))^*(x^*) = \sup_{y\in X}\{\langle y,x^*\rangle + F(y,x)\}.$$

If T is any operator, then $\Phi_{G_T} = \varphi_T$ [1]. Consequently, if $F \in \mathscr{B}_s(X)$, then $\Phi_F = \varphi_{A^F}$ since $F = G_{A^F}$.

Now we show the following useful proposition:

Proposition 3 *Let $A : X \rightrightarrows X^*$ be monotone and $B : X \rightrightarrows X^*$ be maximally monotone and such that φ_B is finite-valued. Consider the following statements:*

1) A is maximally monotone
2) For every $x \in X$, it holds that $R(A + B(\cdot - x)) = X^*$
3) For every $x \in X$, there exists $x' \in X$ such that

$$0 \in A(x) + B(x' - x).$$

Then the following implications hold true: $1) \Rightarrow 2) \Rightarrow 3)$. *If, moreover, B is single-valued and strictly monotone, then these statements are equivalent.*

Proof $1) \Rightarrow 2)$ Let us define $T := B(\cdot - x) = \tau_{(-x,0)}B$. Then T is a monotone operator and by relation (2), φ_T is finite-valued.

By using the first part of the Corollary 2.7 of [9] we obtain that for each $x \in X$, it holds $R(A + B(\cdot - x)) = X^*$.

$2) \Rightarrow 3)$ is obvious.

Now assume that B is also single-valued and strictly monotone. Define B_1 by $B_1(x) = -B(-x)$. It is easy to see that B_1 is maximally monotone, single-valued, strictly monotone, and $\varphi_{B_1}(x,x^*) = \varphi_B(-x,-x^*) < +\infty$. Also, B_1^{-1} is maximally monotone, single valued, strictly monotone and such that $\varphi_{B_1^{-1}}(x^*,x) = \varphi_{B_1}(x,x^*) < +\infty$. From the assumption we infer that for each $x \in X$, $\exists x' \in X$ and $\exists x^* \in X^*$ such that $x^* \in A(x') \cap (-B(x' - x))$, that is, $x^* \in A(x')$ and $x^* \in -B(x' - x) = B_1(x - x')$. This implies $x \in (A^{-1} + B_1^{-1})(x^*)$. Accordingly, $R(A^{-1} + B_1^{-1}) = X$. By using Corollary 2.7 of [9] we deduce that A^{-1} is maximally monotone, hence A is maximally monotone. $\square$

Remark 3 The implication 3) $\Rightarrow$ 1) does not hold in general if the operator B is not single-valued. For example, consider the operators A and B defined on $\mathbb{R}$ by

$$A(x) = \begin{cases} \{1\}, & x > 0 \\ \{0\}, & x = 0 \\ \{-1\}, & x < 0 \end{cases} \qquad B(x) = \begin{cases} \{x+1\}, & x > 0 \\ [-1,1], & x = 0 \\ \{x-1\}, & x < 0 \end{cases}.$$

Then A is monotone but not maximally monotone, B is maximally monotone and strictly monotone, with φ_B finite-valued. To see this, it is enough to show that for given $(x,x^*) \in \mathbb{R}^2$ it holds $\inf_{(y,y^*)\in \text{Graph}(B)}(x-y)(x^*-y^*) > -\infty$. For this to hold, it is enough to have $\inf_{y\in\mathbb{R}\setminus\{0\}}(x-y)(x^*+s-y) > -\infty$, for $s \in \{-1,1\}$. The latter is obviously true since we have the infimum of a quadratic function in y with positive leading coefficient. Hence φ_B is finite-valued. For every $x \in \mathbb{R}$ we set $x' = x$. We see that statement 3) holds since $B(0) = [-1,1]$ while $A(x)$ is one of the sets $\{-1\}$, $\{0\}$ and $\{1\}$. However, statement 1) does not hold.

The equivalence 1) $\Longleftrightarrow$ 3) in Proposition 3 generalizes the following characterization of maximality: a monotone operator A is maximally monotone if and only if for each $x \in X$, there exists $x' \in X$ such that $0 \in J(x'-x)+A(x')$. See for instance [6, p. 324].

3 Main Results

Our first main result gives conditions for a bifunction to be maximally monotone.

Theorem 2 *Let F be a monotone bifunction such that $F(x,\cdot)$ is convex and l.s.c., for each $x \in X$. Then the following statements are equivalent.*

1) *F is an MMB.*
2) *G_{A^F} is a PMMB.*
3) *For every PMMB $H \in \mathscr{B}_m(X)$ such that Φ_H is finite-valued, there exists $x_H \in X$ such that*
$$F(x_H,y)+H(x_H,y) \geq 0, \quad \forall y \in X.$$
4) *There exist a finite-valued $H \in \mathscr{B}_s(X)$ and $p \in X$ such that*
   (a) *$H(p,\cdot)$ is continuous affine,*
   (b) *$H(p,z)+H(z,p) < 0$, $\forall z \in X\setminus\{p\}$,*
   (c) *For every $(x_0,x_0^*) \in X \times X^*$ there exists $\tilde{x} \in X$ satisfying*
$$F(\tilde{x},y)+H(x_0-\tilde{x},x_0-y)-\langle y-\tilde{x},x_0^*\rangle \geq 0, \quad \forall y \in X.$$

Proof 1) $\Rightarrow$ 2) Let F be an MMB. If we set $F_1 = G_{A^F}$, then $F_1 \in \mathscr{B}_m(X)$. By Proposition 1, $A^{F_1} = A^F$, thus F_1 is an MMB. By Proposition 2, F_1 is a PMMB.

2) $\Rightarrow$ 3) Let H be as in statement 3). Setting again $F_1 = G_{A^F}$, by Proposition 2 both F_1 and H are MMB. Since $A^{F_1} = A^F$, the operators A^F and A^H are maximally monotone. In addition, $\varphi_{A^H} = \Phi_H$ is finite-valued.

Let B be the operator defined by $B(x) = -A^H(-x)$. It can be easily seen that B is maximally monotone, and $\varphi_B(x,x^*) = \varphi_{A^H}(-x,-x^*)$. Using implication a) $\Rightarrow$ b) of Theorem 1 we obtain that
$$(0,0) = (w,w^*)+(v,v^*)$$

for some $(w,w^*) \in \text{Graph}(A^F)$ and $(v,v^*) \in \text{Graph}(-B)$. Then $w^* + v^* = 0$, with $w^* \in A^F(w)$, $v^* \in A^H(-v)$ and $w = -v$. Thus,

$$\langle y - w, w^* \rangle \le F(w,y), \quad \forall y \in X$$
$$\langle y - (-v), v^* \rangle \le H(-v,y), \quad \forall y \in X.$$

Adding the two inequalities and setting $x_H = w$ we obtain

$$F(x_H, y) + H(x_H, y) \ge \langle y - x_H, w^* + v^* \rangle = 0, \quad \forall y \in X.$$

3) $\Rightarrow$ 4) We take $p = 0$ and $H = G_J$, where J is the duality mapping. We will show that H satisfies all conditions of 4).

By Proposition 1, $H \in \mathscr{B}_s(X)$. Since

$$H(x,y) = G_J(x,y) = \sup_{x^* \in J(x)} \langle y - x, x^* \rangle$$

and the set $J(x)$ is nonempty and bounded, H is obviously finite-valued. Moreover, for all $y \in X$ the definition of G_J and (1) imply that $G_J(0,y) = 0$ and $G_J(y,0) = -\|y\|^2$. Hence, for $y \ne 0$,

$$H(0,y) + H(y,0) = G_J(0,y) + G_J(y,0) < 0.$$

Given that $G_J(0,\cdot) = 0$, both conditions 4(a) and 4(b) are satisfied. In order to obtain that H satisfies also condition 4(c), given $(x_0, x_0^*) \in X \times X^*$, let us consider $T : X \rightrightarrows X^*$ defined by

$$T(x) = J(x + x_0) - x_0^* = \tau_{(x_0, x_0^*)}(J)(x)$$

and set $\tilde{H} = G_T$. Then T is a maximally monotone operator, so $\tilde{H}$ is a PMMB. We know that $\Phi_{\tilde{H}} = \Phi_{G_T} = \varphi_T$. Hence, by relation (2), $\Phi_{\tilde{H}}$ is finite-valued.

From 3) we obtain that there exists $\tilde{x} \in X$ such that

$$F(\tilde{x}, y) + \tilde{H}(\tilde{x}, y) \ge 0, \quad \forall y \in X. \tag{6}$$

To conclude, let us calculate $\tilde{H}(x,y)$ for $(x,y) \in X \times X^*$. Since $J(-x) = -J(x)$ for all $x \in X$, we obtain $T(x) = -J(x_0 - x) - x_0^*$. It follows that

$$\tilde{H}(x,y) = \sup_{v^* \in T(x)} \langle y - x, v^* \rangle = \sup_{v^* \in -J(x_0 - x) - x_0^*} \langle y - x, v^* \rangle. \tag{7}$$

We set $x^* = -v^* - x_0^*$. Then $v^* \in -J(x_0 - x) - x_0^*$ if and only if $x^* \in J(x_0 - x)$. Hence (7) gives

$$\begin{aligned} \tilde{H}(x,y) &= \sup_{x^* \in J(x_0 - x)} \langle y - x, -x^* - x_0^* \rangle \\ &= \sup_{x^* \in J(x_0 - x)} \langle (x_0 - y) - (x_0 - x), x^* \rangle - \langle y - x, x_0^* \rangle \\ &= G_J(x_0 - x, x_0 - y) - \langle y - x, x_0^* \rangle. \end{aligned}$$

Then (6) becomes

$$F(\tilde{x}, y) + G_J(x_0 - \tilde{x}, x_0 - y) - \langle y - \tilde{x}, x_0^* \rangle \ge 0, \quad \forall y \in X,$$

that is, 4) holds.

4) $\Rightarrow$ 1) Let H and p be as in 4). Since $H(p,\cdot): X \to \mathbb{R}$ is continuous affine, we have that $A^H(p)$ is singleton. By setting $\{p^*\} = A^H(p)$, for every $(y,y^*) \in \mathrm{Graph}(A^H)\setminus\{(p,p^*)\}$, it holds that $y \neq p$ so

$$\langle p-y, p^*-y^*\rangle = -\langle y-p, p^*\rangle - \langle p-y, y^*\rangle \geq -[H(p,y)+H(y,p)] > 0. \tag{8}$$

On the other hand, given $(x_0,x_0^*) \in X \times X^*$, by considering $\tilde{x}$ as in statement 4(c) we have

$$F(\tilde{x},y) + H(x_0-\tilde{x}, x_0-y) - \langle y-\tilde{x}, x_0^*\rangle \geq 0, \ \ \forall y \in X.$$

We define the function g on X by $g(y) = H(x_0-\tilde{x},x_0-y) - \langle y-\tilde{x}, x_0^*\rangle$. Since $H \in \mathscr{B}_s(X)$, g is convex and l.s.c.; it is also finite-valued by assumption 4). Obviously it satisfies $F(\tilde{x},y) + g(y) \geq 0$ for all $y \in X$, and $F(\tilde{x},\tilde{x}) + g(\tilde{x}) = 0$. Hence, $0 \in \partial(F(\tilde{x},\cdot)+g(\cdot))(\tilde{x})$. By the subdifferential sum rule (see for instance [6]), $0 \in \partial F(\tilde{x},\cdot)(\tilde{x}) + \partial g(\cdot)(\tilde{x})$. Hence, there exists $x^* \in X^*$ such that $x^* \in \partial F(\tilde{x},\cdot)(\tilde{x}) = A^F(\tilde{x})$ and $-x^* \in \partial g(\tilde{x})$. The last inclusion yields for every $y \in X$,

$$H(x_0-\tilde{x},x_0-y) - \langle y-\tilde{x}, x_0^*\rangle \geq \langle y-\tilde{x}, -x^*\rangle$$

or

$$H(x_0-\tilde{x},x_0-y) \geq \langle (x_0-y)-(x_0-\tilde{x}), x^*-x_0^*\rangle.$$

Thus, $x^*-x_0^* \in A^H(x_0-\tilde{x})$. Consequently,

$$(x_0,x_0^*) = (\tilde{x},x^*) + (x_0-\tilde{x}, x_0^*-x^*) \in \mathrm{Graph}(A^F) + \mathrm{Graph}(-A^H). \tag{9}$$

Therefore, in view of Remark 1, we have obtained that A^F and A^H satisfy all assumptions (see (8) and (9) above) necessary to conclude that A^F is a maximally monotone operator. This is equivalent to saying that F is an MMB, so 1) holds. □

From the previous results we obtain the following.

Corollary 1 *Let $F: X\times X \to \overline{\mathbb{R}}$ be a monotone bifunction and let $B: X \to X^*$ be a maximally monotone, single-valued and strictly monotone operator such that φ_B is finite-valued. Then F is maximally monotone if, and only if, for every $\lambda > 0$ and for every $x_0 \in X$, there exists $x_\lambda \in X$ such that*

$$\lambda F(x_\lambda,y) + \langle y-x_\lambda, B(x_\lambda-x_0)\rangle \geq 0, \ \forall y \in X. \tag{10}$$

Proof ($\Rightarrow$) Suppose that F is an MMB. Given $x_0 \in X$ fixed, consider the bifunction $H(x,y) = \langle y-x, B(x-x_0)\rangle$. It is clear that $H \in \mathscr{B}_m(X)$. Also, $A^H(x) = \{B(x-x_0)\} = \tau_{(-x_0,0)}B(x)$. Accordingly, A^H is a maximally monotone operator, so H is a PMMB. In addition, from $\Phi_H = \varphi_{A^H} = \varphi_{\tau(-x_0,0)B}$ and relation (2) we deduce that Φ_H is finite-valued.

Set $F_1 = G_{A^F}$. Then $F_1 \in \mathscr{B}_m(X)$ and $A^{F_1} = A^F$, so F_1 is maximally monotone. For each $\lambda > 0$, from the equivalence (2) $\Leftrightarrow$ (3) of the Theorem 2 applied to the bifunctions λF_1 and H, we deduce that there exists $x_\lambda \in X$ such that

$$\lambda F_1(x_\lambda,y) + H(x_\lambda,y) \geq 0, \ \ \forall y \in X,$$

that is,

$$\lambda F_1(x_\lambda,y) + \langle y-x_\lambda, B(x_\lambda-x_0)\rangle \geq 0, \ \ \forall y \in X.$$

On the other hand, it is easy to see that $F_1 \leq F$. Therefore, for each $\lambda > 0$ we obtain that

$$\lambda F(x_\lambda,y) + \langle y-x_\lambda, B(x_\lambda-x_0)\rangle \geq 0, \ \ \forall y \in X.$$

($\Leftarrow$) The inequality (10) is equivalent to $-B(x_\lambda - x) \in A^{\lambda F}(x_\lambda)$. Hence, given $\lambda > 0$, for each $x \in X$ there exists $x_\lambda \in X$ such that $0 \in A^{\lambda F}(x_\lambda) + B(x_\lambda - x)$. By the implication 3) $\Longrightarrow$ 1) of Proposition 3, we have that λF is an MMB, for each $\lambda > 0$ fixed. In particular, F is an MMB. $\square$

Corollary 1 is a generalization of Proposition 2.6 obtained in [5].

We finish by observing that the statements 3) and 4) of the Theorem 2 and the Corollary 1 establish the existence of solutions to equilibrium problems obtained by perturbing the bifunction F according to a choice of a suitable bifunction H for each case.

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Answers to the referees

First of all we would like to express our gratitude to both referees for their careful reading of our manuscript and for their remarks and suggestions.

Regarding the main comments-concerns of referee #1:

1. According to Proposition 1 the operation

$$\mathcal{OM}(X) \ni T \to G_T \in \mathcal{BM}(X)$$

is 1-1 and onto (following the suggestion of the referee, $\mathcal{BM}(X)$ is denoted $\mathcal{B}_m(X)$ in the new version). Hence, *every* element of $\mathcal{B}_m(X)$ is of the form G_T. Thus, indeed the class $\mathcal{B}_m(X)$ is very restrictive. However, an interesting feature of our main results is that they provide characterizations of maximality for more general bifunctions, i.e., monotone bifunctions F who are convex and l.s.c. with respect to the second variable. Thus, the restrictive class $\mathcal{B}_m(X)$ is only a tool for the characterization of a much more general class. In addition, it is interesting that Proposition 2 links maximality of monotone operators to maximality in the sense of the usual order in $\mathbb{R}$.

Pointwise maximal monotone bifunctions are, by definition, elements of $\mathcal{B}_m(X)$, so they are of the form G_T.

2. For any bifunction F, the operator $T := A^F$ always belongs to $\mathcal{O}(X)$ since $A^F(x)$ is closed and convex as it is clear from its definition. Thus, according to Proposition 1, $A^{G_T} = T$. Since $F_1 = G_{A^F} = G_T$ we obtain $A^{F_1} = A^F$.

3. In the new version, we followed the suggestion of the referee.

Regarding the further remarks and recommendations of referee #1:

1. We made the appropriate correction.
2. We followed the referee's suggestion.
3. We followed the referee's suggestion.
4. We added this remark immediately after the definition of $\mathcal{OM}(X)$ ($\mathcal{O}_m(X)$ in the new version).
5. We followed the referee's suggestion.
6. We made the required change.
7. We made the correction.
8. We followed the referee's suggestion. We also used a translation in the proof of Corollary 1.
9. We made the correction.

Regarding the comments of referee #2:

1. We followed the referee's suggestion.
2. We mention the "diagonal subdifferential operators" after the introduction of A^F in page 4.
3. We rephrase Remark 1 to make it more clear.
4. We give such examples in page 4, Remark 2.
5. We finally decided to keep the definition at the same place, because it is the only place where it is used, and we thought that this arrangement would be easier for the reader.