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On a generalization of paramonotone maps and its application to solving the Stampacchia variational inequality

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Paramonotone maps are monotone maps that satisfy a mild additional condition. They were introduced in order to ensure convergence of certain algorithms for solving the Stampacchia variational inequality. Recently, this concept has been generalized to single-valued pseudomonotone$_*$ maps. In this paper we extend this definition to multivalued pseudomonotone$_*$ maps. We show that this new class of maps includes the subdifferentials of locally Lipschitz pseudoconvex functions. In addition, it is shown to be exactly the class of pseudomonotone maps that have a certain cutting plane property, thus ensuring convergence of the above-mentioned algorithms. We give a specific example of a perturbed auxiliary problem method that leads to a solution of the Stampacchia variational inequality with a multivalued, pseudomonotone$_*$ map.

Keywords: Paramonotone maps; Pseudomonotone maps; Variational inequality problems.

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1 Introduction

Let K be a closed convex subset of a Banach space X. A single-valued monotone map $T : K \to X^*$ is called paramonotone if for all $x, y \in K$, $\langle T(x) - T(y), x - y\rangle = 0$ implies that $T(x) = T(y)$. A multivalued monotone map $T : X \to 2^{X^*}$ is called paramonotone if for every $x, y \in X$ and $x^* \in T(x)$, $y^* \in T(y)$, $\langle x^* - y^*, x - y\rangle = 0$ implies that $x^* \in T(y)$ and $y^* \in T(x)$. Maps with the above property were considered for the first time in [1], without being given a name. The name "paramonotone" was introduced in [2].

Paramonotonicity is not a very strong property. It is certainly weaker than strict monotonicity; for instance, the subdifferential of a proper convex lower semicontinuous function is paramonotone [3]. In recent years, paramonotone

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maps have been widely used in algorithms solving variational inequalities [2, 4–6]. The reason is that such maps posess a "cutting plane property" that ensures the convergence of such algorithms (see Section 4 for further details).

Recently, a generalization of paramonotone maps in the single-valued case has been introduced [7], which is related to pseudomonotone, rather than monotone maps. The new maps were called pseudomonotone$_*$ and they also possess the cutting plane property. In addition, the gradient of a differentiable pseudoconvex function is pseudomonotone$_*$. These maps were also considered in [8] in connection with the maximum principle sufficiency property.

In this paper, the notion of pseudomonotone$_*$ maps is extended to the multivalued case. We show that this extension is natural from various points of view: it is related to the newly introduced notions of D-maximal pseudomonotone maps and cyclically pseudomonotone maps in a way that is entirely analogous to the relation between paramonotone, maximal monotone and cyclically monotone maps; the subdifferential of a locally Lipschitz pseudoconvex function is pseudomonotone$_*$; these maps not only possess the cutting plane property, but more importantly they are also characterized by this property in a sense that will be made precise. Thus, it seems that the class of pseudomonotone$_*$ maps is exactly the class which is suitable for many algorithms based on the cutting plane property.

As an illustration, we show in the last section how pseudomonotone$_*$ maps can be applied to a perturbed auxiliary problem method in order to solve the Stampacchia variational inequality.

We start by giving some notation and definitions. In the following X will be a Banach space, X^* its dual and $\langle x^*, x\rangle$ the duality product of $x^* \in X^*$ with $x \in X$. For any subset A of X^*, we denote by $\mathbb{R}_{++}A$ the set

$$\mathbb{R}_{++}A = \{kx^* : x^* \in A, k \in \mathbb{R}_{++}\}$$

where $\mathbb{R}_{++} = (0, +\infty)$. For a locally Lipschitz function $f : X \to \mathbb{R} \cup \{+\infty\}$ we will denote by $\partial^o f$ the Clarke subdifferential of f [9]. A locally Lipschitz function f is called pseudoconvex, if for all $x, y \in X$ and $x^* \in \partial^o f(x)$ the following implication holds:

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow f(y) \geq f(x).$$

The graph of a multivalued map $T : X \to 2^{X^*}$ will be denoted by $\operatorname{Gr} T$. For the definitions and the classic results on monotone, maximal monotone and cyclically monotone maps we refer to [10]. We recall that a map T is called pseudomonotone on $K \subseteq X$ [11] if for every $x, y \in K$ and $x^* \in T(x)$,

$y^* \in T(y)$,

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

Equivalently, T is pseudomonotone on K if or every $x, y \in K$ and $x^* \in T(x)$, $y^* \in T(y)$,

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle > 0.$$

The map T is called strictly pseudomonotone on K if for every $x, y \in K$ $(x \neq y)$ and $x^* \in T(x)$, $y^* \in T(y)$,

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle > 0.$$

2 Single-valued pseudomonotone$_*$ maps

In this section we restrict our attention to single-valued maps. Single-valued pseudomonotone$_*$ maps have been defined as a generalization of single-valued paramonotone maps.

Definition 2.1 [7] A single-valued map $T : K \to X^*$ is called pseudomonotone$_*$ if it is pseudomonotone and for all $x, y \in K$, $\langle T(x), y - x\rangle = \langle T(y), y - x\rangle = 0$ implies that $T(y) = kT(x)$ for some $k > 0$.

One would probably expect to call such maps "para-pseudomonotone". However we decided to follow [7] in calling such maps pseudomonotone$_*$, not only because this name is now established [8], but also because the prefix "para" is somehow misleading: these maps are more than pseudomonotone, not less.

Equivalently, a map T is pseudomonotone$_*$ if for all $x, y \in K$, the following implication holds:

$$\langle T(x), y - x\rangle \geq 0 \Rightarrow \begin{cases} \langle T(y), y - x\rangle > 0 \\ \text{or} \\ T(y) = kT(x) \text{ for some } k > 0. \end{cases} \tag{1}$$

Indeed, let T be pseudomonotone$_*$, and assume that $\langle T(x), y - x\rangle \geq 0$. If $\langle T(x), y - x\rangle > 0$, then pseudomonotonicity gives $\langle T(y), y - x\rangle > 0$. If $\langle T(x), y - x\rangle = 0$, then the definition of pseudomonotonicity$_*$ implies immediately that $\langle T(y), y - x\rangle > 0$, unless $T(y) = kT(x)$ for some $k > 0$. Conversely, suppose that a map has property (1). If $\langle T(x), y - x\rangle \geq 0$, then this property implies that either $\langle T(y), y - x\rangle > 0$, or $T(y) = kT(x)$ for some $k > 0$, in which case $\langle T(y), y - x\rangle = k \langle T(x), y - x\rangle \geq 0$; thus, T is pseudomonotone. Pseudomonotonicity$_*$ now follows trivially.

It was shown in [7] that for a differentiable and pseudoconvex function $f : K \to \mathbb{R}$, ∇f is pseudomonotone$_*$.

It is easy to see that between the various notions we encountered so far, the following implications hold in the single-valued case. Here, "st." stands for "strictly". All implications are one-sided.

$$\begin{array}{ccccc} \text{st. pseudomonotone} & \Rightarrow & \text{pseudomonotone}_* & \Rightarrow & \text{pseudomonotone} \\ \Uparrow & & \Uparrow & & \Uparrow \\ \text{st. monotone} & \Rightarrow & \text{paramonotone} & \Rightarrow & \text{monotone} \end{array} \tag{2}$$

3 Multivalued pseudomonotone$_*$ maps

Let $T : X \to 2^{X^*}$ be a multivalued map. If T is the subdifferential of a proper lsc convex function, then T is paramonotone [2]. In preparation of what is to follow, we give another proof of the same result:

PROPOSITION 3.1 *If $T : X \to 2^{X^*}$ is cyclically monotone and maximal monotone, then it is paramonotone.*

Proof Assume that $x, y \in X$ and $x^* \in T(x), y^* \in T(y)$ are such that $\langle x^* - y^*, x - y\rangle = 0$. For every $z \in X$, $z^* \in T(z)$, by cyclic monotonicity,

$$\langle x^*, z - x\rangle + \langle y^*, x - y\rangle + \langle z^*, y - z\rangle \leq 0.$$

Since $\langle y^*, x - y\rangle = \langle x^*, x - y\rangle$, we get

$$\begin{aligned} -\langle x^* - z^*, y - z\rangle &= \langle x^*, z - x\rangle + \langle x^*, x - y\rangle + \langle z^*, y - z\rangle \\ &= \langle x^*, z - x\rangle + \langle y^*, x - y\rangle + \langle z^*, y - z\rangle \leq 0. \end{aligned}$$

This is true for all $(z, z^*) \in \mathrm{Gr}(T)$. Since T is maximal monotone, we infer that $x^* \in T(y)$. By symmetry, we also get $y^* \in T(x)$. □

Since the subdifferential of a proper lsc convex function is both maximal monotone and cyclically monotone [10], we deduce:

COROLLARY 3.2 *The subdifferential of a proper lsc convex function is a paramonotone map.*

Note that the above results were already mentioned without proof in a 30 years old paper of Bruck [1].

A definition of pseudomonotone$_*$ multi-valued maps was given in [7]. There, a multivalued map $T : X \to 2^{X^*}$ is called pseudomonotone$_*$ if it is pseudomonotone and for every $x, y \in X$ and $x^* \in T(x)$, $y^* \in T(y)$, $\langle x^*, y - x\rangle = \langle y^*, y - x\rangle = 0$ imply that there exists $k > 0$ such that $ky^* \in T(x)$. This definition is an obvious generalization of the definitions of multivalued paramonotone maps and single-valued pseudomonotone$_*$ maps. However, unlike in the differentiable case, there is no known theorem that relates multivalued pseudomonotone$_*$ maps to subdifferentials of (nonsmooth) pseudoconvex functions. Thus this definition, in spite of its attractiveness, does not seem to be supported by results.

In order to introduce an appropriate definition for multivalued pseudomonotone$_*$ maps, we recall some definitions and results from [12]. Let $T : X \to 2^{X^*}$ be pseudomonotone. We denote by $D(T)$ the domain of T and by Z_T the set of zeros of T, i.e., $Z_T = \{x \in X : 0 \in T(x)\}$. If $S : X \to 2^{X^*}$ is also pseudomonotone, we say that T and S are equivalent and write $S \sim T$ if the following hold:

(i) $D(T) = D(S)$,

(ii) $Z_T = Z_S$,

(iii) For all $x \notin Z_T$, $\mathbb{R}_{++}T(x) = \mathbb{R}_{++}S(x)$.

Thus, T and S are equivalent if they have the same domain, the same set of zeros, and for each x that is not a zero, every element of $T(x)$ is a positive multiple of an element of $S(x)$ and vice-versa. This equivalence is very closely related to the Stampachia variational inequality problem. We recall that given a subset $K \subseteq X$ and a multivalued map T, the Stampachia variational inequality problem is the following:

$$\text{find } x \in K \text{ such that } \exists x^* \in T(x) : \forall y \in K, \langle x^*, y - x\rangle \geq 0. \tag{3}$$

Let $S(T, K)$ be the set of its solutions. The following proposition holds [13]:

Proposition 3.3 *Let T_1, T_2 be pseudomonotone maps. If $T_1 \sim T_2$, then $S(T_1, K) = S(T_2, K)$ for every convex subset K of X. Conversely, if $S(T_1, K) = S(T_2, K)$ for every convex subset K of X and the maps T_1, T_2 have w^*-compact convex values, then $T_1 \sim T_2$.*

The relation $\sim$ is an equivalence relation, thus dividing the set of all pseudomonotone maps into equivalence classes. Given a pseudomonotone map T, its equivalence class has a maximum with respect to graph inclusion, denoted by $\widehat{T}$. We can get an idea about how to construct $\widehat{T}$ if we remember that it has to be equivalent to T and at the same time be "as large as possible". If $x \in D(T)$ is not a zero of T, then it is clear that we should take $\widehat{T}(x) = \mathbb{R}_{++}T(x)$ (i.e., the cone generated by $T(x)$, without the origin). If $x \in Z_T$, then we should adjoin to $T(x)$ as many elements as possible, without destroying pseudomonotonicity. We do this as follows. First, we note that whenever $x \in Z_T$, pseudomonotonicity combined with $0 \in T(x)$ imply

$$\forall y \in D(T), \forall y^* \in T(y), \langle y^*, y - x \rangle \geq 0.$$

If $y \in D(T)$ is such that for at least one $y^* \in T(y)$, $\langle y^*, y - x \rangle = 0$ holds, then we know that if we are to adjoin an element x^* to $T(x)$ without destroying pseudomonotonicity, then $\langle x^*, x - y \rangle \geq 0$ has to hold. Let $L_{T,x}$ be the set of all such y, i.e.,

$$L_{T,x} = \{y \in D(T) : \exists y^* \in T(y) : \langle y^*, y - x \rangle = 0\}.$$

Let further

$$N_{L_{T,x}} = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \forall y \in L_{T,x}\}$$

be the normal cone to $L_{T,x}$. According to what we just explained, $\widehat{T}$ should be defined as

$$\widehat{T}(x) = \begin{cases} N_{L_{T,x}}, \text{ if } x \in Z_T, \\ \mathbb{R}_{++}T(x), \text{ if } x \notin Z_T \end{cases}.$$

Indeed, it can be shown that the map thus defined is the maximum of the equivalence class of T [12]. We are now ready for the definition of pseudomonotone$_*$ maps:

Definition 3.4 A map T is called pseudomonotone$_*$ on K if it is pseudomonotone and for every $x, y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$, $\langle x^*, x - y \rangle = \langle y^*, x - y \rangle = 0$ implies that $x^* \in \widehat{T}(y)$ and $y^* \in \widehat{T}(x)$.

Note that in order to show that a pseudomonotone map is pseudomonotone$_*$, it is sufficient to show that $x^* \in T(x)$, $y^* \in T(y)$ and $\langle x^*, x - y \rangle = \langle y^*, x - y \rangle = 0$ imply that $y^* \in \widehat{T}(x)$; due to symmetry, we can deduce that $x^* \in \widehat{T}(y)$.

It is easy to see that an alternative analogous to (1) holds: a pseudomonotone map T is pseudomonotone$_*$ on K if and only if for any $x, y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$, the following implication holds:

$$\langle x^*, y - x \rangle \geq 0 \Rightarrow \begin{cases} \langle y^*, y - x \rangle > 0 \\ \text{or} \\ x^* \in \widehat{T}(y) \text{ and } y^* \in \widehat{T}(x). \end{cases}$$

The definition above differs from the definition given in [7] in the points

$x, y \in Z_T$. Indeed, whenever $y \notin Z_T$, then $x^* \in \widehat{T}(y)$ is the same as $kx^* \in T(y)$ for some $k > 0$.

We check that this definition is compatible with the corresponding definition for single-valued maps.

PROPOSITION 3.5 *Definitions 2.1 and 3.4 are equivalent for single-valued maps $T : K \to X^*$.*

Proof Let T be pseudomonotone$_*$ according to the Definition 3.4, and assume that for some $x, y \in K$, $\langle T(x), y - x\rangle = \langle T(y), y - x\rangle = 0$ holds. If $T(x) \neq 0$, then $\widehat{T}(x) = \mathbb{R}_{++}T(x)$. By Definition 3.4, $T(y) \in \widehat{T}(x)$. Thus there exists $k > 0$ such that $T(y) = kT(x)$. If on the contrary $T(x) = 0$, then again by Definition 3.4 we deduce that $0 \in \widehat{T}(y)$. It follows that necessarily $T(y) = 0$. Hence we have $T(y) = kT(x)$ with $k = 1$.

The converse is obvious. □

It is easy to see that a paramonotone map is pseudomonotone$_*$, and a strictly pseudomonotone map is pseudomonotone$_*$. Thus, the diagram (2) holds also for multivalued maps, all implications being one-sided.

As one could expect from what happens with paramonotonicity, there exists a relation between pseudomonotonicity$_*$, cyclic pseudomonotonicity and a kind of maximal pseudomonotonicity. We first give the relevant definitions.

Definition 3.6 [12] A pseudomonotone map T is called D-maximal pseudomonotone if $\widehat{T}$ has no pseudomonotone extension with the same domain, apart from itself.

An equivalent, more apealing definition of D-maximal pseudomonotonicity is given by the following proposition:

PROPOSITION 3.7 *[13] A pseudomonotone map T with convex domain is D-maximal pseudomonotone if and only if each pseudomonotone extension of T with the same domain, is equivalent to T.*

Finally, we mention the following result:

PROPOSITION 3.8 *[12] The Clarke subdifferential $\partial^o f$ of a locally Lipschitz pseudoconvex function f is a D-maximal pseudomonotone map.*

We also recall the notion of cyclic pseudomonotonicity [14, 15]: a map T is called cyclically pseudomonotone if for every $x_i \in X$ and $x_i^* \in T(x_i)$, $i = 1, 2, \ldots, n$, the following implication holds:

$$\langle x_i^*, x_{i+1} - x_i\rangle \geq 0 \text{ for all } i = 1, 2, \ldots, n-1 \Rightarrow \langle x_n^*, x_1 - x_n\rangle \leq 0.$$

We now have:

PROPOSITION 3.9 *If T is a D-maximal pseudomonotone, cyclically pseudomonotone map with convex domain, then T is pseudomonotone$_*$.*

Proof Let $(x, x^*) \in \mathrm{Gr}\,(T)$ and $(y, y^*) \in \mathrm{Gr}\,(T)$ be such that $\langle x^*, y - x\rangle = \langle y^*, y - x\rangle = 0$. It is enough to show that the map T_1 defined by

$$T_1\,(z) = \begin{cases} T\,(z)\,,\ z \neq x \\ T\,(x) \cup \{y^*\}\,,\ z = x \end{cases}$$

is pseudomonotone; indeed, by Proposition 3.7 this would mean that T_1 is equivalent to T. Since $\widehat{T}$ is the maximum element of the equivalence class of T, one has $T_1\,(x) \subseteq \widehat{T}\,(x)$ which implies that $y^* \in \widehat{T}\,(x)$.

In order to show that T_1 is pseudomonotone, it is enough to show the following implications, for each $z \in X$, $z^* \in T\,(z)$:

$$\langle z^*, x - z\rangle \geq 0 \Rightarrow \langle y^*, x - z\rangle \geq 0 \tag{4}$$

$$\langle y^*, z - x\rangle \geq 0 \Rightarrow \langle z^*, z - x\rangle \geq 0. \tag{5}$$

Suppose that $\langle z^*, x - z\rangle \geq 0$. Then $\langle x^*, y - x\rangle = 0$ and cyclic pseudomonotonicity imply that $\langle y^*, z - y\rangle \leq 0$. But $\langle y^*, x - z\rangle = \langle y^*, x - y\rangle + \langle y^*, y - z\rangle \geq 0$, hence (4) holds.

Suppose that $\langle y^*, z - x\rangle \geq 0$; then $\langle y^*, z - y\rangle = \langle y^*, z - x\rangle + \langle y^*, x - y\rangle \geq 0$. Combining with $\langle x^*, y - x\rangle = 0$ and using cyclic pseudomonotonicity on the points $x_1 = x$, $x_2 = y$ and $x_3 = z$, we get $\langle z^*, x - z\rangle \leq 0$, i.e., (5) holds. □

As mentioned before, if f is a differentiable pseudoconvex function, then f' is pseudomonotone$_*$. Proposition 3.9 allows an immediate generalization to the nonsmooth case:

PROPOSITION 3.10 *A locally Lipschitz function $f : X \to X^* \cup \{+\infty\}$ is pseudoconvex if and only if its Clarke subdifferential $\partial^o f$ is pseudomonotone$_*$.*

Proof If $f : X \to X^* \cup \{+\infty\}$ is a locally Lipschitz pseudoconvex function, then it is known that $\partial^o f$ is D-maximal pseudomonotone [12] and cyclically pseudomonotone [14]; in addition, its domain is convex (see Proposition 11.4 in [11]). Thus, Proposition 3.9 applies and shows that $\partial^o f$ is pseudomonotone$_*$. □

The class of pseudomonotone$_*$ maps is relatively vast. For instance, it is easy to see that for a single-valued paramonotone (or, more generally, pseudomonotone$_*$) map T and any positive function $f : X \to \mathbb{R}$ the map $T_1(\cdot) = f(\cdot)T(\cdot)$ is pseudomonotone$_*$. More generally, the following holds:

PROPOSITION 3.11 *If $T : X \to 2^{X^*}$ is pseudomonotone$_*$ and S is a pseudomonotone map equivalent to T, then S is pseudomonotone$_*$.*

Proof Let $x, y \in K$ and (x, x^*), $(y, y^*) \in \operatorname{Gr} S$ be such that $\langle x^*, y - x\rangle = \langle y^*, y - x\rangle = 0$. Assume first that $x, y \notin Z_S$. Then there exist $k > 0$, $l > 0$ and $x_1^* \in T(x)$, $y_1^* \in T(y)$ such that $x^* = kx_1^*$ and $y^* = ly_1^*$. Hence $\langle x_1^*, y - x\rangle = \langle y_1^*, y - x\rangle = 0$. Using that T is pseudomonotone$_*$ we deduce that $x_1^* \in \widehat{T}(y)$. Since S and T are equivalent and the maximum element of their equivalence class is unique, we immediately deduce that $x^* \in \widehat{S}(y)$.

We now show that if $x \in Z_S$, then $y \in Z_S$. Indeed, assume that this is not the case. Then there exist $k > 0$ and $y_1 \in T(y)$ such that $y^* = ky_1^*$. Since $x \in Z_S = Z_T$ and T is pseudomonotone$_*$, we deduce from $\langle 0, x - y\rangle = \langle y_1^*, x - y\rangle$ that $0 \in \widehat{T}(y)$, hence $0 \in T(y)$, a contradiction.

Thus the only remaining case is $x \in Z_S$ and $y \in Z_S$. For any $z \in L_{S,y}$ let $z^* \in S(z)$ be such that $\langle z^*, y - z\rangle = 0$. Since $0 \in T(y)$ and $\langle 0, y - z\rangle = 0$, by applying what we just showed to the points y, z we deduce that $0 \in S(z)$. By pseudomonotonicity, $\langle x^*, x - z\rangle \geq 0$. Thus, $\langle x^*, z - y\rangle = \langle x^*, z - x\rangle + \langle x^*, x - y\rangle \leq 0$. It follows that $x^* \in N_{L_{S,y}} = \widehat{S}(y)$.

Thus in all cases $x^* \in \widehat{S}(y)$, and S is pseudomonotone$_*$. □

4 The cutting plane property

The main reason for studying paramonotone maps is a property which is very useful in constructing algorithms for solving variational inequality problems.This property runs as follows:

$$\left.\begin{array}{c} x \in S(T, K) \\ y \in K \\ \langle y^*, x - y\rangle \geq 0 \text{ for some } y^* \in T(y) \end{array}\right\} \Rightarrow y \in S(T, K). \tag{6}$$

Property (6) says the following: assuming that the Stampacchia variational inequality has solutions, if at the nth iteration of an algorithm we find a point y_n that is not a solution of the Stampacchia variational inequality, then we know that all solutions are contained in the intersection of K with the open halfspace $\{x \in X : \langle y_n^*, x - y_n\rangle < 0\}$ where y_n^* is an arbitrary element of $T(y_n)$.

In [2,4] it was shown that paramonotone maps have this property. As we will see, the same property holds for pseudomonotone$_*$ maps. In addition, in some sense that will be made precise, it characterizes these maps in the class of all pseudomonotone maps. Thus the class of pseudomonotone$_*$ maps as defined in the previous section is exactly the suitable class for algorithms using the cutting plane property (6).

THEOREM 4.1 *Let T be pseudomonotone on a convex set K.*

(i) If T is pseudomonotone$_*$, then property (6) holds on every subset of K.

(ii) Conversely, if T has convex, w^*-compact values and has property (6) on every convex, compact subset of K, then T is pseudomonotone$_*$ on $\operatorname{int} K$.

Proof (i) Suppose that T is pseudomonotone$_*$ on K, $K_1 \subseteq K$, $x \in S(T, K_1)$, $y \in K_1$ and there exists $y^* \in T(y)$ such that $\langle y^*, x - y \rangle \geq 0$. Since $x \in S(T, K_1)$, there exists $x^* \in T(x)$ such that $\langle x^*, z - x \rangle \geq 0$ for all $z \in K_1$. In particular, $\langle x^*, y - x \rangle \geq 0$. By pseudomonotonicity, $\langle y^*, y - x \rangle \geq 0$. Also, from the given relation $\langle y^*, x - y \rangle \geq 0$ we get $\langle x^*, x - y \rangle \geq 0$, thus $\langle x^*, y - x \rangle = \langle y^*, y - x \rangle = 0$. Using pseudomonotonicity$_*$ we infer that $x^* \in \widehat{T}(y)$. We consider two cases:

(a) If $0 \in T(y)$, then obviously $y \in S(T, K_1)$, and we are done.

(b) If $0 \notin T(y)$, then there exists $y_1^* \in T(y)$ and $k > 0$ such that $y_1^* = kx^*$. For every $z \in K_1$,

$$\langle y_1^*, z - y \rangle = k\left(\langle x^*, z - x \rangle + \langle x^*, x - y \rangle\right) = k\langle x^*, z - x \rangle \geq 0.$$

Thus $y \in S(T, K_1)$ and property (6) holds on K_1.

(ii) Suppose that T has convex w^*-compact values and that (6) holds on each convex compact subset of K. Let $y \in K$, $x \in \operatorname{int} K$ and $x^* \in T(x)$, $y^* \in T(y)$ be given such that $\langle x^*, y - x \rangle = \langle y^*, y - x \rangle = 0$. We want to show that $y^* \in \widehat{T}(x)$. We again consider two cases:

(a) If $0 \in T(x)$, then $x \in S(T, K)$. For every $z \in L_{T,x}$ there exists $z^* \in T(z)$ such that $\langle z^*, z - x \rangle = 0$. Consider the compact convex set $K_1 = \operatorname{co}\{x, z, y\}$. Since $x \in S(T, K_1)$, from property (6) we infer that $z \in S(T, K_1)$. Thus there exists $z_1^* \in T(z)$ such that $\langle z_1^*, y - z \rangle \geq 0$. By pseudomonotonicity, $\langle y^*, y - z \rangle \geq 0$. It follows that

$$\langle y^*, z - x \rangle = \langle y^*, z - y \rangle + \langle y^*, y - x \rangle = \langle y^*, z - y \rangle \leq 0.$$

This means that $y^* \in N_{L_{T,x}} = \widehat{T}(x)$.

(b) If $0 \notin T(x)$, then we have to show that $y^* \in \mathbb{R}_{++}T(x)$. Suppose that $y^* \notin \mathbb{R}_{++}T(x)$; then $T(x) \cap \mathbb{R}_+ y^* = \emptyset$. By the Hahn-Banach Theorem, there exists $v \in X$ such that $\langle ty^*, v \rangle > \langle z^*, v \rangle$ for all $t \geq 0$ and $z^* \in T(x)$. It follows that $\langle y^*, v \rangle \geq 0 > \langle z^*, v \rangle$ for all $z^* \in T(x)$. Choose $t > 0$ small enough so that the point $w = x + tv$ belongs to K. Then $K_2 := \operatorname{co}\{w, x, y\} \subseteq K$, and for each $z \in K_2$, $z = t_1 x + t_2 y + t_3 w$ with t_1, t_2, t_3 non-negative and $t_1 + t_2 + t_3 = 1$ one has:

$$\langle y^*, z - y \rangle = t_1\langle y^*, x - y \rangle + t_3\langle y^*, w - y \rangle = t_3\left(\langle y^*, x - y \rangle + \langle y^*, tv \rangle\right) \geq 0.$$

Thus $y \in S(T, K_2)$. However, for every $z^* \in T(x)$,

$$\langle z^*, w - x \rangle = t \langle z^*, v \rangle < 0,$$

thus $x \notin S(T, K_2)$. Hence property (6) does not hold on the set K_2, a contradiction. □

The following example shows that in case (ii) of the above theorem we cannot expect that T is pseudomonotone$_*$ on the whole set K.

Example 4.2 Set $X = \mathbb{R}^3$, $K = [0,1] \times \{0\} \times \{0\}$ and define $T : K \to \mathbb{R}^3$ by $T(x,0,0) = (0, x, 1-x)$, $x \in [0,1]$. Then $S(T,K) = K$. Hence property (6) holds on every subset of K. For any distinct $x, y \in [0,1]$, we have

$$\langle T(x,0,0), (y,0,0) - (x,0,0) \rangle = \langle T(y,0,0), (y,0,0) - (x,0,0) \rangle = 0.$$

However, it is obvious that $T(y,0,0) \neq kT(x,0,0)$ for all $k > 0$. Thus T is not pseudomonotone$_*$.

It is worth noticing that in case T is any map which is single-valued and hemicontinuous (that is, continuous on line segments of K with respect to the w*-topology on X^*), property (6) implies pseudomonotonicity of T on K. Thus pseudomonotonicity$_*$ is in this sense equivalent to property (6).

Proposition 4.3 *Let $T : K \to X^*$ be hemicontinuous. If T has property (6) on every convex compact subset of K, then it is pseudomonotone on K, and pseudomonotone$_*$ on int$K$.*

Proof Suppose that T is not pseudomonotone. Then there exist $x, y \in K$ such that $\langle T(x), y - x \rangle \geq 0$ and $\langle T(y), y - x \rangle < 0$. Since T is hemicontinuous, we can find $z \in [x, y)$ such that $\langle T(z), y - x \rangle = 0$ and also $w \in [z, y)$ such that $\langle T(w), x - y \rangle > 0$. Then we can verify that $z \in S(T, [x,y])$, $\langle T(w), z - w \rangle > 0$ and $w \notin S(T, [x,y])$. Thus, property (6) does not hold. It follows that T is pseudomonotone. The second assertion of the proposition follows from Theorem 4.1. □

5 Application to a perturbed auxiliary problem method

Paramonotone maps have been widely used in algorithms for the solution of variational inequalities. Most of the methods and proofs also work for pseudomonotone$_*$ maps. We will present one of them which is an adaptation of the method used for instance in [16] and [17].

Let K be a closed convex subset of a Hilbert space H and $T : K \to 2^H$ a multivalued map. Consider any Gâteaux differentiable strongly convex function $M : H \to \mathbb{R}$. One usually takes $M(x) = \|x\|^2/2$. Given an arbitrary $\bar{x} \in K$ and $\mu > 0$, we consider the perturbed problem

$$\begin{aligned} &\text{find } x \in K \text{ such that } \exists x^* \in T(x) : \\ &\forall y \in K, \quad \left\langle \mu x^* + M'(x) - M'(\bar{x}), y - x \right\rangle \geq 0. \end{aligned} \tag{7}$$

This problem may be much easier than the original problem (3). Actually it is a variational inequality problem for the map $T_1(\cdot) = \mu T(\cdot) + M'(\cdot) - M'(\bar{x})$. If for instance T is weakly monotone on K and μ is small enough, then T_1 is strongly monotone on K [17].

A sequence $(x_k)_{k\in\mathbb{N}}$ will be constructed by the following algorithm (see also [16] for a slightly different algorithm and [17] for a version with a single-valued map T):

Algorithm 1 (i) Choose an arbitrary $x_0 \in K$.
(ii) Having chosen x_k, find $x_{k+1} \in K$ and $x^*_{k+1} \in T(x_{k+1})$ so that

$$\forall y \in K, \quad \left\langle \mu_k x^*_{k+1} + M'(x_{k+1}) - M'(x_k), y - x_{k+1} \right\rangle \geq 0. \tag{8}$$

In what follows, we assume that both (3) and the auxiliary problems (8) have solutions. Given a fixed solution x of (3), we define the Lyapunov function $\Phi_x(\cdot)$ as follows.

$$\Phi_x(y) = M(x) - M(y) - \left\langle M'(y), x - y \right\rangle.$$

By strong convexity of M we get immediately

$$\Phi_x(y) \geq (\beta/2)\|y - x\|^2, \quad \text{for all } y \in H \tag{9}$$

where $\beta > 0$ is the strong convexity modulus of M.

In the proof below, we follow mainly [17], while paying attention to the fact that H is not finite-dimensional.

PROPOSITION 5.1 *Let $K \subseteq H$ be convex and $T : K \to 2^H$ be a pseudomonotone map. Let further $M : H \to \mathbb{R}$ be a strongly convex function with strong convexity modulus β. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence defined by Algorithm 1 where $\mu_k \geq a > 0$. Finally, let $x \in K$ be a solution of (3). Then the sequence $(\Phi_x(x_k))_{k\in\mathbb{N}}$ is nonincreasing and converging, $\|x_{k+1} - x_k\| \to 0$, the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded, and* $\lim_{k\to+\infty} \langle x^*_k, x_k - x\rangle = 0$.

Proof From the definition of Φ_x by rearranging terms and using (9) and (8) we obtain

$$\begin{aligned}\Phi_x(x_k)-\Phi_x(x_{k+1}) &= M(x_{k+1})-M(x_k)-\left\langle M'(x_k),x_{k+1}-x_k\right\rangle \\ &\quad +\left\langle M'(x_{k+1})-M'(x_k),x-x_{k+1}\right\rangle \\ &\geq \frac{\beta}{2}\left\|x_{k+1}-x_k\right\|^2+\mu_k\left\langle x_{k+1}^*,x_{k+1}-x\right\rangle . \end{aligned} \tag{10}$$

Since x is a solution of (3), pseudomonotonicity of T implies $\left\langle x_{k+1}^*, x_{k+1}-x\right\rangle \geq 0$, hence $\Phi_x(x_k)-\Phi_x(x_{k+1})\geq 0$.

Thus $\Phi_x(x_k)$ is nonincreasing. Since $\Phi_x \geq 0$, $\Phi_x(x_k)$ converges. From (9) it follows that the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded. From (10) we see that $\lim_{k\to+\infty}\|x_{k+1}-x_k\| = 0$ and $\lim_{k\to+\infty}\mu_k\left\langle x_{k+1}^*,x_{k+1}-x\right\rangle = 0$. Using $\mu_k\geq a>0$ we obtain $\lim_{k\to+\infty}\left\langle x_{k+1}^*,x_{k+1}-x\right\rangle=0$. □

In order to proceed further, we consider the following continuity condition on T (see also [16]).

Condition (A) *For any bounded sequence* $(y_k)_{k\in\mathbb{N}}$ *and any* $y_k^*\in T(y_k)$, *there exists a subsequence* $(y_{k'})_{k'\in N_1\subseteq\mathbb{N}}$ *such that the limits* $\bar{y} = w-\lim_{k'\to+\infty}y_{k'}$ *and* $\bar{y}^* = w^*-\lim_{k'\to+\infty}y_{k'}^*$ *exist,* $\bar{y}^*\in T(\bar{y})$, *and* $\lim_{k'\to+\infty}\langle y_{k'}^*,y_{k'}\rangle \geq \langle \bar{y}^*,\bar{y}\rangle$.

Condition (A) is automatically true if T is a single-valued continuous map and H is finite-dimensional. Also, if H is finite-dimensional and T is multivalued with compact convex values and upper semicontinuous on straight lines, then there exists an equivalent map S with compact convex values which is upper semicontinuous (see Theorem 3.7 of [12]). Thus, in this case we can replace T by S for which Condition (A) holds automatically.

PROPOSITION 5.2 *Assumptions as in Proposition 5.1. Further assume that* T *is pseudomonotone*$_*$ *and that Condition (A) holds. Then every weak cluster point of* $(x_k)_{k\in\mathbb{N}}$ *is a solution of (3). If in addition* M' *is weakly continuous, then the whole sequence* $(x_k)_{k\in\mathbb{N}}$ *converges weakly to a solution of (3).*

Proof Let $(x_{k'})_{k'\in N_1\subseteq\mathbb{N}}$, be any subsequence of $(x_k)_{k\in\mathbb{N}}$ weakly converging to $\bar{x}$. Using Condition (A), we deduce that there exists a subsequence $(x_{k''})_{k''\in N_2\subseteq\mathbb{N}_1}$ such that $x_{k''}\to\bar{x}$ weakly, while $x_{k''}^*$ weak*-converges to some $\bar{x}^*\in T(\bar{x})$ and $\lim_{k''\to+\infty}\left\langle x_{k''}^*,x_{k''}\right\rangle\geq\langle\bar{x}^*,\bar{x}\rangle$. Using Proposition 5.1 we infer that

$$\langle\bar{x}^*,x\rangle = \lim_{k''\to+\infty}\left\langle x_{k''}^*,x\right\rangle = \lim_{k''\to+\infty}\left\langle x_{k''}^*,x_{k''}\right\rangle \geq \langle\bar{x}^*,\bar{x}\rangle .$$

Thus, $\langle\bar{x}^*,x-\bar{x}\rangle\geq 0$. Since T is pseudomonotone$_*$ and x is a solution, $\bar{x}$ is also a solution.

The rest of the proof is rather standard [16, 17]: assume that $\bar{x}$ and $\hat{x}$ are two cluster points, limits of two subsequences $(x_{k'})$ and $(x_{k''})$ of (x_k). By what we just proved, both $\bar{x}$ and $\hat{x}$ are solutions of (3). By Proposition 5.1, $\Phi_{\bar{x}}(x_k)$ and $\Phi_{\hat{x}}(x_k)$ converge. Let $\bar{l}$ and $\hat{l}$ be their respective limits. Since

$$\Phi_{\hat{x}}(x_{k'}) - \Phi_{\bar{x}}(x_{k'}) = M(\hat{x}) - M(\bar{x}) - \left\langle M'(x_{k'}), \hat{x} - \bar{x} \right\rangle, \tag{11}$$

by taking limits we obtain $\hat{l} - \bar{l} = M(\hat{x}) - M(\bar{x}) - \langle M'(\bar{x}), \hat{x} - \bar{x} \rangle$. Similarly, using the subsequence $(x_{k''})$ in (11) we obtain $\hat{l} - \bar{l} = M(\hat{x}) - M(\bar{x}) - \langle M'(\hat{x}), \hat{x} - \bar{x} \rangle$. Thus, $\langle M'(\hat{x}) - M'(\bar{x}), \hat{x} - \bar{x} \rangle = 0$. From the strong monotonicity of M' we infer that $\bar{x} = \hat{x}$, as desired. □

We note that the function $M(y) = \|y\|^2/2$ in a Hilbert space is strongly convex and $M'(y) = y$ is weakly continuous. Thus this function is a suitable candidate for using in the above results.

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