Invited Paper

Pseudomonotone Operators: a Survey of the Theory and its Applications

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Abstract. The notion of pseudomonotone operator in the sense of Karamardian has been studied for 35 years and has found many applications in variational inequalities and economics. The purpose of this survey paper is to present the most fundamental results in this field, starting from the earliest developments and reaching the latest results and some open questions. The exposition includes: the relation of (generally multivalued) pseudomonotone operators to pseudoconvex functions; first-order characterizations of single-valued, differentiable pseudomonotone operators; application to variational inequalities; the notion of equivalence of pseudomonotone operators and its application to maximality; a generalization of paramonotonicity and its relation to the cutting-plane method; and the relation to the revealed preference problem of mathematical economics.

Key Words. Pseudomonotone operators, variational inequalities, pseudomonotone$_*$ operators

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1 Introduction

Pseudomonotone operators were introduced back in 1976 by Karamardian [1] as a generalization of monotone operators. Actually, before Karamardian, pseudomonotonicity has been used by economists to describe a property of a consumer's demand correspondence. But, for a long time, mathematicians working on generalized monotonicity notions largely ignored the corresponding developments in economics.

In the 35 years following Karamardian's definition, the theory of pseudomonotone operators has been considerably developed and found many applications. The notion has been generalized to multivalued operators, and applied to variational inequalities. Variational inequalities were a source of inspiration for further developments: appropriate generalizations were introduced to study vector variational inequalities and pseudomonotone equilibrium problems. Other lines of research include finding first-order characterizations of pseudomonotone operators, establishing the relation to generalized convex functions, solving the revealed preference problem in economics, etc.

This paper contains a survey of the main results in the theory and applications of pseudomonotone operators, starting by the older ones and reaching the frontiers of the subject. The proofs of many of these results are contained in research papers, and in this case we will only give the reference. We will start by fixing the notation and recalling some definitions in next section. We will then proceed by giving some examples of pseudomonotone operators, and their link to pseudoconvex functions. Also, we will give first-order characterizations. Section 4 is devoted to variational inequalities with pseudomonotone operators. We will see that a notion of equivalence of pseudomonotone operators arises quite naturally. In Section 5, we will see how this equivalence gives rise to the notion of maximal pseudomonotonicity. This in its turn will lead in Section 6, to the introduction of a particular class of pseudomonotone operators, that are exactly those suitable to the cutting plane method for the algorithmic solution of the variational inequality. Section 7 is devoted to the revealed preference problem in economics and its relation to some very recent developments in the theory of pseudomonotone operators. The last section contains a selection of open questions.

Some generalizations will be deliberately omitted: Vector variational inequalities and the corresponding pseudomonotonicity notion; pseudomonotone bifunctions; pseudomonotonicity in the sense of Brezis, etc. We will also omit almost all other generalized monotonicity notions, unless they are necessary to the exposition of the theory of pseudomonotone operators.

2 Notation and Definitions

In what follows, X will be a Banach space and X^* its topological dual. For $x, y \in X$ we set $[x, y] = \{tx + (1-t)y : t \in [0,1]\}$. Given a multivalued operator $T : X \rightrightarrows X^*$, we denote by $D(T)$ its domain and by $\mathrm{gr}(T)$ its graph.

Also, we will denote by Z_T the set of its zeros, i.e., $Z_T := T^{-1}\{0\}$. For the upper semicontinuity, lower semicontinuity and other properties of multivalued operators, the reader is referred to any standard book on nonlinear analysis such as [2]. We denote $]0, +\infty[$ by $\mathbb{R}_{++}$. Given a set A in a Banach space, we denote by $\operatorname{conv} A$ its convex hull, $\operatorname{int} A$ its interior, and we set $\mathbb{R}_{++}A := \cup_{t>0} tA$. For a proper, lower semi-continuous (lsc) convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ we denote by ∂f the subdifferential of f in the sense of convex analysis.

A multivalued operator $T : X \rightrightarrows X^*$ is called:

1. monotone, iff for every (x, x^*), $(y, y^*) \in \operatorname{gr}(T)$,
$$\langle y^* - x^*, y - x\rangle \geq 0;$$
2. pseudomonotone, iff for every (x, x^*), $(y, y^*) \in \operatorname{gr}(T)$ the following implication holds:
$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0;$$
3. quasimonotone, iff for every (x, x^*), $(y, y^*) \in \operatorname{gr}(T)$ the following implication holds:
$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

It is clear that every monotone operator is pseudomonotone, and every pseudomonotone operator is quasimonotone.

There is an important difference between monotone and pseudomonotone operators, that should be mentioned. If T_1 and T_2 are two monotone operators, then $T_1 + T_2$ is monotone. For pseudomonotone operators, this does not hold: the sum of pseudomonotone operators is not pseudomonotone in general. In fact, if T is such that $T + x^*$ is quasimonotone (or pseudomonotone) for all $x^* \in X^*$, then T is monotone; see [3, Prop. 2.1], or [4] for a stronger result. On the other hand, if T is any operator and $f : D(T) \to]0, +\infty[$ is any function, then $f(\cdot)T(\cdot)$ is pseudomonotone, if and only if T is pseudomonotone. This property does not hold for monotone operators, and will be the starting point for the important notion of equivalence in Section 4.

Another, stronger property than pseudomonotonicity is cyclic pseudomonotonicity. We recall that an operator T is called cyclically monotone iff for every finite sequence $x_1, x_2, \ldots, x_n$ in X and any choice $x_i^* \in T(x_i)$ one has:
$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i\rangle \leq 0,$$
where we set $x_{n+1} = x_1$. It is known that the subdifferential ∂f of a proper, lsc convex function f is cyclically monotone and, conversely, for any cyclically monotone operator T, there exists a proper, lsc convex function f such that $\operatorname{gr}(T) \subseteq \operatorname{gr}(\partial f)$.

An operator T will be called cyclically pseudomonotone iff for every finite sequence $x_1, x_2, \ldots, x_n$ and any choice $x_i^* \in T(x_i)$, $i = 1, \ldots n$ the following implication holds:
$$\langle x_i^*, x_{i+1} - x_i\rangle \geq 0 \text{ for all } i = 1, 2, \ldots n-1 \Rightarrow \langle x_n^*, x_1 - x_n\rangle \leq 0$$

By considering a sequence consisting of two elements, one sees that any cyclically pseudomonotone operator is pseudomonotone. Obviously, a cyclically monotone operator is cyclically pseudomonotone.

3 Examples and Characterizations of Pseudomonotone Operators

The simplest class of pseudomonotone operators consists of the gradients of pseudoconvex functions. Given an open convex set $C \subseteq \mathbb{R}^n$, we recall that a differentiable function $f : C \to \mathbb{R}$ is called pseudoconvex iff for every $x, y \in C$, the following implication holds:

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x).$$

Karamardian [1] has shown that a differentiable function f is pseudoconvex if and only if its gradient ∇f is pseudomonotone. This result has been generalized in many ways for nonsmooth functions and multivalued operators. For instance, pseudoconvexity may be defined by using the Clarke-Rockafellar subdifferential, as follows. We first recall that given a lsc function f, one defines the Clarke-Rockafellar directional derivative at $x_0 \in \operatorname{dom} f$ in the direction $d \in X$ by

$$f^{\uparrow}(x_0, d) := \sup_{\varepsilon > 0} \limsup_{x \to_f x_0, t \searrow 0} \inf_{\|d' - d\| \leq \varepsilon} \frac{f(x + td') - f(x)}{t};$$

Here, $x \to_f x_0$ means that $x \to x_0$ and $f(x) \to f(x_0)$. Then one defines the Clarke-Rockafellar subdifferential of f at x_0 by

$$\partial^{\uparrow} f(x_0) := \left\{ x^* \in X^* : \langle x^*, d \rangle \leq f^{\uparrow}(x_0, d), \forall d \in X \right\}.$$

For $x_0 \notin \operatorname{dom} f$ one sets $\partial^{\uparrow} f(x_0) = \emptyset$.

Definition 3.1 *A proper lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called pseudoconvex* [5] *iff for every $x, y \in X$ the following implication holds:*

$$\exists x^* \in \partial^{\uparrow} f(x) : \langle x^*, y - x \rangle \geq 0 \Longrightarrow f(x) \leq f(y).$$

In case f be a proper lsc convex function, $\partial^{\uparrow} f$ is equal to the subdifferential ∂f of convex analysis (see Theorem 5 in [6]), hence

$$\forall x^* \in \partial f(x) : \langle x^*, y - x \rangle \leq f(y) - f(x).$$

Consequently, a proper lsc convex function is pseudoconvex.

A proper lsc function is convex, if and only if $\partial^{\uparrow} f$ is monotone [7]. The following result of [5] (see [8] for the cyclically pseudomonotone part) relates pseudoconvexity of a function to pseudomonotonicity of its subdifferential.

Theorem 3.1 *Let f be lsc and radially continuous. Then the following are equivalent:*

(i) f is pseudoconvex;

(ii) $\partial^{\uparrow} f$ is pseudomonotone;

(iii) $\partial^{\uparrow} f$ is cyclically pseudomonotone.

Note that, whenever f is pseudoconvex, lsc and radially continuous, it is also quasiconvex ([5], Corollary 3.1), hence it is actually continuous ([9], Proposition 9).

Of course, there are many examples of pseudomonotone operators that are not subdifferentials. As we remarked before, if T is any monotone or pseudomonotone operator and $f : X \to]0, +\infty[$ is any function, then the operator T_1 defined by $T_1(x) := f(x)T(x)$ is pseudomonotone.

Other examples stem from the following characterization of single-valued differentiable pseudomonotone operators. This deep result is the final fruit of a series of papers by Crouzeix-Ferland, John, and Brighi. See [10] for a relatively simple proof.

Theorem 3.2 *Let $C \subseteq \mathbb{R}^n$ be an open convex set and $T : C \to \mathbb{R}^n$ be continuously differentiable. Then T is pseudomonotone, if and only if the following two conditions hold:*

$$x \in C, \langle T(x), h\rangle = 0 \Rightarrow \langle T'(x)h, h\rangle \geq 0 \tag{1}$$

$$\left.\begin{array}{c} x \in C, T(x) = 0 \\ T'(x)h = 0 \end{array}\right\} \Rightarrow \forall \bar{t} > 0, \exists t \in]0, \bar{t}] : \langle T(x + th), h\rangle \geq 0. \tag{2}$$

We single out two special cases, where condition (1) alone is sufficient to characterize pseudomonotonicity of an operator T in an open convex subset C of $\mathbb{R}^n$. Whenever T is differentiable on C and T is never zero, then obviously (2) is automatically satisfied so (1) is a necessary and sufficient condition for T to be pseudomonotone. Another case is when T is affine, i.e., has the form $T(x) = Mx + q$ where M is an $n \times n$ matrix and $q \in \mathbb{R}^n$. Then $T' = M$ so whenever $T(x) = 0$ and $T'(x)h = 0$ hold, one has $T(x+th) = Mx+q+tMh = 0$. Hence condition (2) is again satisfied. Condition (1) becomes

$$x \in C, \langle Mx + q, h\rangle = 0 \Rightarrow \langle Mh, h\rangle \geq 0. \tag{3}$$

Consequently, condition (3) is necessary and sufficient for T to be pseudomonotone on C. This result was found initially in [11].

4 Variational Inequalities and Equivalence of Pseudomonotone Operators

Given a nonempty subset K of X and an operator T, the Stampacchia Variational Inequality (VI) is to find $x \in K$ such that

$$\forall y \in K, \exists x^* \in T(x) : \quad \langle x^*, y - x\rangle \geq 0. \tag{VI}$$

Variational inequalities have found many applications in optimization and in other applied fields, especially in mechanics.

In the special case where the values of T on K are nonempty, convex and weak*-compact, it is a consequence of the minimax theorem that every solution x of VI is actually a solution of the strong Stampacchia variational inequality: find $x \in K$ such that

$$\exists x^* \in T(x) : \forall y \in K, \quad \langle x^*, y - x \rangle \geq 0. \tag{SVI}$$

In the case of pseudomonotone operators, VI can be transformed to another, equivalent problem, the so-called Minty Variational Inequality (MVI) consisting in finding $x \in K$ such that

$$\forall y \in K, \forall y^* \in T(x) : \quad \langle y^*, y - x \rangle \geq 0. \tag{MVI}$$

We call T upper sign-continuous at $x \in D(T)$ iff for all $v \in X$, the following implication holds:

$$\inf_{x^* \in T(x+tv)} \langle x^*, v \rangle \geq 0, \ \forall t \in]0, 1[\Rightarrow \sup_{x^* \in T(x)} \langle x^*, v \rangle \geq 0.$$

Upper sign-continuity is a very weak notion of continuity. For instance, any operator whose restriction on straight lines of its domain is upper semicontinuous (usc) with respect to the weak*-topology on X^*, is upper sign-continuous. Any positive function on $\mathbb{R}$ is upper sign-continuous.

The following proposition relates MVI to VI. Let $S(T, K)$ be the set of solutions of VI, i.e., the set of all $x \in K$ satisfying (VI), and let $S_M(T, K)$ be the set of solutions of MVI. The proof is well-known; see for instance [12] for a version of it.

Proposition 4.1 *If T is pseudomonotone, then $S(T, K) \subseteq S_M(T, K)$. If T is upper sign-continuous, then $S_M(T, K) \subseteq S(T, K)$.*

Due to Proposition 4.1, in order to solve VI for a pseudomonotone, upper sign-continuous operator, it is enough to solve MVI. A proof for the existence of solutions of VI first appeared in [13] for single-valued pseudomonotone operators, then in [14] for multi-valued pseudomonotone operators. Here we present a version from [15], where it was shown that actually quasimonotonicity is enough. Let $S_{str}(T, K)$ be the set of solutions of (SVI).

Theorem 4.1 *Let K be a nonempty, closed and convex subset of a reflexive Banach space X and $T : X \rightrightarrows X^*$ be a quasimonotone operator. Assume that T is upper sign-continuous with nonempty, convex, weakly compact values on K, and that the following coercivity condition is satisfied:*

$$\exists \rho > 0 : \forall x \in K, \|x\| > \rho, \exists z \in K, \|z\| < \|x\| : \forall x^* \in T(x), \langle x^*, x - z \rangle \geq 0.$$

Then $S_{str}(T, K) \neq \emptyset$.

For quasimonotone operators, it is not true that $S(T,K) = S_M(T,K)$. In fact, under the assumptions of Theorem 4.1, one has $S_M(T,K) \subseteq S(T,K)$. Note that $S_M(T,K)$ might be empty as shown by concrete examples [16]. However, whenever K is weakly compact, it can be shown that $x \in S(T,K)$ if and only if x is *locally* a solution of the MVI. This is the key of the proof of Theorem 4.1.

If T is an operator, say single-valued, and $g : X \to]0, +\infty[$ a function, then we already remarked that T is pseudomonotone if and only if the operator $T_1(x) = g(x)T(x)$ is pseudomonotone. Further, we note that $x \in S(T,K)$ if and only if $x \in S(T_1,K)$ for all convex subsets K. Thus, from the point of view of variational inequalities, T and T_1 are indistinguishable. One can develop this idea further by introducing the notion of equivalence of arbitrary pseudomonotone operators [17]:

Definition 4.1 *Two pseudomonotone operators T_1 and T_2 are equivalent iff the following conditions are satisfied:*
(i) $Z_{T_1} = Z_{T_2}$;
(ii) $\mathbb{R}_{++}T_1(x) = \mathbb{R}_{++}T_2(x)$ for all $x \in X \backslash Z_{T_1}$.

This means that T_1 and T_2 should have the same set of zeros, and for every x that is not a zero, every element of $T_1(x)$ is a positive multiple of an element of $T_2(x)$ and vice versa. We denote the equivalence of T_1 and T_2 by $T_1 \sim T_2$. Note that condition (ii) in Definition 4.1 implies that T_1 and T_2 have the same domain: if $x \notin D(T_1)$ then $\emptyset = \mathbb{R}_{++}T_1(x) = \mathbb{R}_{++}T_2(x)$; thus, $x \notin D(T_2)$.

The term "equivalence" is justified by the fact that, under some weak conditions, two pseudomonotone operators are equivalent if and only if the corresponding VI has the same solutions, as the following theorem shows.

Theorem 4.2 *Let T_1 and T_2 be two pseudomonotone operators. If $T_1 \sim T_2$, then $S(T_1,K) = S(T_2,K)$ for all convex subsets K of X. Conversely, if $S(T_1,K) = S(T_2,K)$ for all straight line segments K of X, and T_1, T_2 have convex, weak*-compact values, then $T_1 \sim T_2$.*

Proof. Assume first that $T_1 \sim T_2$. Let K be any convex set. We show that $S(T_1,K) \subseteq S(T_2,K)$. Let $x \in S(T_1,K)$. If $x \in Z_{T_1}$, then $x \in Z_{T_2}$ and this obviously implies that $x \in S(T_2,K)$. So assume that $x \notin Z_{T_1}$. Since $x \in S(T_1,K)$, for every $y \in K$ there exists $x^* \in T_1(x)$ such that $\langle x^*, y - x\rangle \geq 0$. But $T_1(x) \subseteq \mathbb{R}_{++}T_2(x)$ by assumption, so there exists $t > 0$ such that $tx^* \in T_2(x)$. Since $\langle tx^*, y - x\rangle \geq 0$, we obtain $x \in S(T_2,K)$, that is, $S(T_1,K) \subseteq S(T_2,K)$. Likewise, we obtain the reverse inclusion, consequently $S(T_1,K) = S(T_2,K)$.

Now assume that $S(T_1,[a,b]) = S(T_2,[a,b])$ holds for all line segments $[a,b]$, and that T_1, T_2 have convex, weak*-compact values. To show equality (i) in Definition 4.1, assume that $x \in Z_{T_1}$ but $x \notin Z_{T_2}$. Then $0 \notin T_2(x)$ and by the separation theorem, there exists $v \in X$ such that for all $x^* \in T_2(x)$, one has $\langle x^*, v\rangle < 0$. The last inequality can be written as $\langle x^*, (x+v) - x\rangle < 0$ for all $x^* \in T_2(x)$, hence it implies that $x \notin S(T_2, [x, x+v])$. On the other hand, since by assumption $0 \in T_1(x)$, we infer that $x \in S(T_1, [x, x+v])$ thus arriving to a contradiction. Hence, (i) holds.

To show condition (ii), let us show first that $D(T_1) = D(T_2)$. Indeed, for each $x \in D(T_1)$ one has $S(T_2, [x, x]) = S(T_1, [x, x]) = \{x\} \neq \emptyset$, so $x \in D(T_2)$, hence $D(T_1) \subseteq D(T_2)$ and by symmetry we have equality. Consequently, we need to show condition (ii) only for $x \in D(T_1) \backslash Z_{T_1}$. Now assume that there exists $x \in D(T_1) \backslash Z_{T_1} = D(T_2) \backslash Z_{T_2}$ such that $\mathbb{R}_{++} T_1(x) \neq \mathbb{R}_{++} T_2(x)$, say $\mathbb{R}_{++} T_1(x) \not\subseteq \mathbb{R}_{++} T_2(x)$. Then there exists $x^* \in T_1(x)$ such that $x^* \notin \mathbb{R}_{++} T_2(x)$. It follows that the sets $\mathbb{R}_+\{x^*\} := \{tx^* : t \geq 0\}$ and $T_2(x)$ have an empty intersection. By the separation theorem, there exists $v \in X$ such that for all $z^* \in T_2(x)$ and all $t \geq 0$, $\langle tx^*, v\rangle > \langle z^*, v\rangle$. It follows easily that $\langle z^*, v\rangle < 0$ for all $z^* \in T_2(x)$, and $\langle x^*, v\rangle \geq 0$. These imply that $x \notin S(T_2, [x, x+v])$ and $x \in S(T_1, [x, x+v])$, thus contradicting condition (ii). $\square$

The notion of equivalence is a key tool both in the theory and applications of pseudomonotone operators, as we will see in the next sections.

5 Maximal Pseudomonotone Operators

We recall that a monotone operator T is called maximal monotone iff it has no monotone extension except for itself; in other words, if S is a monotone operator such that $\mathrm{gr}(T) \subseteq \mathrm{gr}(S)$, then $S = T$. A somewhat weaker notion is the following: A monotone operator T is called D-maximal monotone iff it has no monotone extension with the same domain, except for itself; that is, $\mathrm{gr}(T) \subseteq \mathrm{gr}(S)$ and $D(T) = D(S)$ imply $T = S$.

Maximal monotone operators have many nice properties, such as:

P1 If T is monotone, then it has a maximal monotone extension.

P2 If T is D-maximal monotone, then for each $x \in X$, $T(x)$ is weak*-closed and convex.

P3 If T is monotone, upper hemicontinuous on $D(T)$ with weak*-closed convex values and $D(T)$ is open, then T is D-maximal monotone. In particular, T is usc on $D(T)$.

P4 The subdifferential of a lsc proper convex function is maximal monotone.

P5 A D-maximal monotone operator is usc at every point of $\operatorname{int} D(T)$.

These properties are related with some properties of all monotone operators:

P6 If T is monotone and lsc at $x_0 \in \operatorname{int} D(T)$, then T is single-valued at x_0.

P7 If T satisfies the Aubin property[4] around a point (x, x^*) of its graph, then T is single-valued for all x in a neighborhood of x.

See for instance [2] for properties P1-P6 and [18] for P7. If one tries to transcribe these properties to pseudomonotone operators in a straightforward way, one can see immediately that almost all of them do not hold. For instance, by analogy to Property P4 above, one would expect that the subdifferential of a pseudoconvex function is maximal pseudomonotone; however, if f is a continuously differentiable pseudoconvex function, then one can easily check that the operator $T(x) = f'(x)$ is pseudomonotone, but it has a nontrivial pseudomonotone extension, namely $T_1(x) = \mathbb{R}_{++}\{f'(x)\}$. Hence, maximal pseudomonotonicity has to be defined differently for Property P4 to hold. Also, the operator $T : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $T(x) :=]0, +\infty[$ for all $x \in \mathbb{R}$ is pseudomonotone and lsc, but not single-valued, which is not what one would expect in view of Property P6.

A solution to this problem is provided by the notion of equivalence. As we will see, in most cases, a result of the form "if T is monotone, then it has property A" becomes "if T is pseudomonotone, then there exists an equivalent operator T that has property A". This is most welcome in VI, because we can replace the initially given operator by another one that has better properties, and still find the same solutions; see [19, Corol. 11] for a concrete application of this idea to an algorithm solving a VI.

Given a pseudomonotone operator T, let $[T]$ be its equivalence class:

$$[T] := \{S : X \rightrightarrows X^* : S \text{ is pseudomonotone, } S \sim T\}.$$

It is easy to check that the operator $\widehat{T}$ defined by $\widehat{T}(x) := \bigcup_{S\in[T]} S(x)$ is pseudomonotone and is equivalent to T. It is of course the maximum element of $[T]$, with respect to graph inclusion. The following proposition gives an explicit construction for $\widehat{T}$.

Proposition 5.1 *The operator $\widehat{T}$ is given by the formula:*

$$\widehat{T}(x) = \begin{cases} N_{L_{T,x}}, & \text{if } x \in Z_T \\ \mathbb{R}_{++}T(x), & \text{if } x \notin Z_T \end{cases} \tag{4}$$

where $L_{T,x}$ is the set

$$L_{T,x} := \{y \in D(T) : \exists y^* \in T(y) : \langle y^*, y - x\rangle = 0\}$$

and $N_{L_{T,x}}$ is the normal cone to $L_{T,x}$ at x:

$$N_{L_{T,x}} := \{x^* \in X^* : \langle x^*, y - x\rangle \le 0, \forall y \in L_{T,x}\}.$$

[4]Given two Banach spaces X and Y, an operator $T : X \rightrightarrows Y$ is said to satisfy the Aubin property around $(x, y) \in \operatorname{gr} T$ iff there exist neighbourhoods $V$of x, U of y and a positive real number l such that

$$T(v) \cap U \subseteq T(v') + l\|v' - v\|\overline{B}_Y(0, 1), \quad \forall\, v, v' \in V$$

where $\overline{B}_Y(0, 1)$ denotes the closed unit ball of Y.

Proof. Let us call T_1 the operator defined by the right hand-side of (4). We first show that T_1 is pseudomonotone. Let $(x,x^*),(y,y^*)\in \mathrm{gr}(T_1)$ be such that $\langle x^*, y-x\rangle \geq 0$. Note that there exists $x_1^*\in T(x)$ such that $\langle x_1^*, y-x\rangle \geq 0$; in fact, if $x\in Z_T$ then we can take $x_1^*=0$, otherwise $x^*\in \mathbb{R}_{++}T(x)$, so we can take $x_1^*=tx^*$ for some $t>0$. Since T is pseudomonotone, it follows that $\langle z^*, y-x\rangle \geq 0$ for all $z^*\in T(y)$. If $y\notin Z_T$, then obviously $\langle z^*, y-x\rangle \geq 0$ for all $z^*\in T_1(y)=\mathbb{R}_{++}T(y)$. If $y\in Z_T$, then again pseudomonotonicity of T implies that $\langle x_1^*, x-y\rangle \geq 0$. Consequently, $\langle x_1^*, x-y\rangle = 0$ and $x\in L_{T,y}$. Hence for any $z^*\in T_1(y)=N_{L_{T,y}}$ one has $\langle z^*, y-x\rangle \geq 0$, and T_1 is pseudomonotone.

It is clear that T_1 is equivalent to T. Let $S\in[T]$. For all $x\notin Z_T$ one has $S(x)\subseteq \mathbb{R}_{++}S(x)=\mathbb{R}_{++}T(x)=T_1(x)$. Now let $x\in Z_T$. We have to show that $S(x)\subseteq N_{L_{T,x}}$, i.e., for each $x^*\in S(x)$ and $y\in L_{T,x}$, $\langle x^*, x-y\rangle \geq 0$ holds. We consider two cases: if $y\in Z_S=Z_T$, then $\langle 0, x-y\rangle = 0$ together with the pseudomonotonicity of S, imply that $\langle x^*, x-y\rangle \geq 0$ for all $x^*\in S(x)$. In the second case, assume that $y\notin Z_S$. By definition of $L_{T,x}$, there exists $y^*\in T(y)$ such that $\langle y^*, y-x\rangle = 0$. Since $\mathbb{R}_{++}T(y)=\mathbb{R}_{++}S(y)$, there exists $y_1^*\in S(y)$ such that $\langle y^*, x-y\rangle = 0$ holds; then we deduce again $\langle x^*, x-y\rangle \geq 0$ for all $x^*\in S(x)$. Thus, $S(x)\subseteq T_1(x)$ in all cases, so T_1 is the maximum element of $[T]$, i.e., it is equal to $\widehat{T}$. □

Note that the definition of $\widehat{T}$ does not exclude that $\widehat{T}$ has a pseudomonotone extension with the same domain; it only says that it does not have a pseudomonotone extension with the same domain, which is equivalent to T. We are now ready to define maximal pseudomonotonicity:

Definition 5.1 *A pseudomonotone operator T is called D-maximal pseudomonotone iff $\widehat{T}$ has no pseudomonotone extension with the same domain except for itself.*

There is a practical way to check maximal pseudomonotonicity, provided by the following lemma. See [17] for the proof.

Lemma 5.1 *Let T be a pseudomonotone operator. Suppose that for any $(x,x^*)\in (D(T)\backslash Z_T)\times X^*$ such that $\{(x,x^*)\}\cup \mathrm{gr}(T)$ is the graph of a pseudomonotone operator, one has $x^*\in \mathbb{R}_{++}T(x)$. Then T is D-maximal pseudomonotone. The converse is also true, provided that $D(T)$ is convex.*

As a first application, we give the following result on the set of zeros Z_T of a D-maximal pseudomonotone operator. If T is pseudomonotone and $x\in Z_T$, then it is clear that for any $(y,y^*)\in \mathrm{gr}(T)$, one has $\langle y^*, y-x\rangle \geq 0$. In fact, this is true for all $(y,y^*)\in \mathrm{gr}(\widehat{T})$ since $x\in Z_{\widehat{T}}$. It is interesting that for D-maximal pseudomonotone operators, the converse also holds:

Proposition 5.2 *Let T be a D-maximal pseudomonotone operator and $x\in D(T)$. Then $x\in Z_T$, if and only if*

$$\forall\,(y,y^*)\in \mathrm{gr}(\widehat{T}),\quad \langle y^*, y-x\rangle \geq 0. \tag{5}$$

If in addition $D(T)$ is convex, then $x \in Z_T$ if and only if

$$\forall\, (y, y^*) \in \mathrm{gr}(T), \quad \langle y^*, y - x\rangle \geq 0. \tag{6}$$

Proof. If (5) holds, then for all $y^* \in \widehat{T}(y)$ the following two implications are true:

$$\langle 0, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0$$
$$\langle y^*, x - y\rangle \geq 0 \Rightarrow \langle 0, x - y\rangle \geq 0.$$

It follows that the operator with graph $\mathrm{gr}(\widehat{T}) \cup \{(x, 0)\}$ is pseudomonotone. Since $\widehat{T}$ is D-maximal pseudomonotone, we deduce that $0 \in T(x)$, i.e., $x \in Z_T$.

Now assume that $D(T)$ is convex and (6) holds. With the same argument as before, $\mathrm{gr}(T) \cup \{(x, 0)\}$ is the graph of a pseudomonotone operator. If we suppose that $x \notin Z_T$, then it follows from Lemma 5.1 that $0 \in \mathbb{R}_{++}T(x)$ thus $0 \in T(x)$, a contradiction. $\square$

In view of the previous proposition, the set of zeros is given by

$$Z_T = \bigcap_{(y,y^*)\in \mathrm{gr}(\widehat{T})} \{x \in D(T) : \langle y^*, y - x\rangle \geq 0. \tag{7}$$

Hence, Z_T is the intersection of $D(T)$ with a family of closed halfspaces. We deduce the following corollary:

Corollary 5.1 *If T is a D-maximal pseudomonotone operator, then Z_T is closed in $D(T)$. If in addition $D(T)$ is convex, then Z_T is convex.*

We now are in position to show that, in case $D(T)$ is convex, there is another equivalent, but more attractive definition of D-maximal pseudomonotonicity.

Proposition 5.3 *Let T be pseudomonotone with convex domain. Then T is D-maximal pseudomonotone if and only if every pseudomonotone extension of T with the same domain is equivalent to T.*

Proof. Let T be D-maximal pseudomonotone and let S be a pseudomonotone extension of T with $D(S) = D(T)$. We show first that $Z_T = Z_S$. Clearly, $Z_T \subseteq Z_S$ since S is an extension of T. If $x \in Z_S$, then pseudomonotonicity of S implies that for every $(y, y^*) \in \mathrm{gr}(S)$, $\langle y^*, y - x\rangle \geq 0$. This holds *a fortiori* for every $(y, y^*) \in \mathrm{gr}(T)$, so $x \in Z_T$ by Proposition 5.2, hence $Z_T = Z_S$.

Given $x \in D(T)\backslash Z_T$ one has obviously $\mathbb{R}_{++}T(x) \subseteq \mathbb{R}_{++}S(x)$. Now take any $x^* \in S(x)$. The operator with graph $\mathrm{gr}(T) \cup \{(x, x^*)\}$ is a restriction of S, thus it is pseudomonotone. Since T is D-maximal monotone, Lemma 5.1 implies that $x^* \in \mathbb{R}_{++}T(x)$. Accordingly, $\mathbb{R}_{++}T(x) = \mathbb{R}_{++}S(x)$ so $S \sim T$.

Conversely, assume that T is a pseudomonotone operator such that every pseudomonotone extension of T with the same domain is equivalent to T. If S is a pseudomonotone extension of $\widehat{T}$ with the same domain, then S is also an extension of T, so $S \sim T$. Since $\widehat{T}$ is the largest element of $[T]$, it follows that $S = \widehat{T}$. This means that T is D-maximal monotone. $\square$

Let us check if properties P1-P7 of monotone operators can be modified to hold for pseudomonotone ones. Exactly as for monotone operators, an application of Zorn's lemma shows that Property P1 of monotone operators has its pseudomonotone counterpart:

Proposition 5.4 *Every pseudomonotone operator T has a D-maximal pseudomonotone extension with the same domain.*

Property P2 of maximal monotone operators is only partially recovered. In fact, let T be D-maximal pseudomonotone. Then $\widehat{T}$ does not admit a pseudomonotone extension except for itself. However, it is easy to see that the operator $\operatorname{conv}\widehat{T}$ defined by $(\operatorname{conv}\widehat{T})(x) := \operatorname{conv}(\widehat{T}(x))$ is pseudomonotone. Hence $\operatorname{conv}\widehat{T} = \widehat{T}$ and we deduce the following property:

Proposition 5.5 *If T is D-maximal pseudomonotone, then $\widehat{T}(x)$ is convex for all $x \in X$.*

It is not true that $\widehat{T}(x)$ is weak*-closed, since for $x \notin Z_T$, $\widehat{T}(x)$ is a cone without 0. It is not even true that $\widehat{T}(x) \cup \{0\}$ is weak*-closed, unless some additional conditions are met (see, e.g., [17]).

Property P3 also has its pseudomonotone counterpart [17]:

Proposition 5.6 *Let T be pseudomonotone and upper sign-continuous, $D(T)$ be open and $T(x)$ be weak*-compact and convex for all $x \in D(T)$. Then T is $D-$maximal pseudomonotone.*

If f is a locally Lipschitz function, then $\operatorname{dom}(f)$ is open, $D\left(\partial^{\uparrow} f\right) = \operatorname{dom}(f)$, $\partial^{\uparrow} f(x)$ is weak*-compact and convex for all $x \in D\left(\partial^{\uparrow} f\right)$, and $\partial^{\uparrow} f$ is upper semicontinuous in the strong-to-weak* topology [20]. In particular, $\partial^{\uparrow} f$ is upper sign-continuous. If in addition f is pseudoconvex, then $\partial^{\uparrow} f$ is pseudomonotone. Hence, Proposition 5.6 entails the following version of Property P4 for pseudomonotone operators:

Corollary 5.2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a pseudoconvex, locally Lipschitz function. Then $\partial^{\uparrow} f$ is a $D-$maximal pseudomonotone operator.*

Property P5 cannot be recovered as it is. It is not true that for each D-maximal pseudomonotone operator, there exists an equivalent pseudomonotone operator that is upper semicontinuous (or even upper sign-continuous) at the interior of its domain. However, in finite dimensions it can be shown (see [17]) that this is true in a form similar to the one given in the second part of P3:

Theorem 5.1 *Let $T : \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ be pseudomonotone, upper sign-continuous on $D(T)$, with compact convex values. Suppose that $D(T)$ is an open convex set. Then there exists a pseudomonotone operator T_1 with compact convex values which is usc on $D(T)$, and such that $T_1 \sim T$.*

Regarding properties P6 and P7, we have the following proposition for the pseudomonotone case. Let us first define two operators T_1 and T_2 to be locally equivalent at $x_0 \in X$, iff there exists a neighborhood U of x_0 such that the restrictions of T_1 and T_2 on U are equivalent.

Proposition 5.7 *Let T be a pseudomonotone operator and $x_0 \in \operatorname{int} D(T)$.*

(i) If T is lower semicontinuous at x_0, then $T(x) = \{0\}$ or $T(x) \subseteq \mathbb{R}_{++}\{x_0^*\}$ for some $x_0^* \in T(x_0)$.

(ii) If T is lower semicontinuous in a neighborhood of $x_0 \notin Z_T$, then T is locally equivalent around x_0 to a single-valued, norm-to-weak* continuous operator.

(iii) In case X is finite-dimensional, assertion (ii) is true without the assumption $x_0 \notin Z_T$.

Proof. Part (i) was proved in Prop. 3.9 of [17]. To prove part (ii), we remark that by Proposition 5.2, we know that there exists $(y, y^*) \in \operatorname{gr}(\widehat{T})$ such that $\langle y^*, x_0 - y\rangle > 0$. Let U be an open neighborhood of x_0 such that $U \subseteq D(T)$, T is lower semicontinuous on U, and $\langle y^*, x - y\rangle > 0$ for all $x \in U$. By pseudomonotonicity, we obtain $\langle x^*, x - y\rangle > 0$ for all $x^* \in T(x)$, $x \in U$. Obviously, U does not intersect Z_T. For each $x \in U$, select $F(x) \in \mathbb{R}_{++}T(x)$ such that $\langle F(x), x - y\rangle = 1$. By part (i), we know that $\mathbb{R}_{++}T(x) = \mathbb{R}_{++}\{F(x)\}$, so T is locally equivalent to the single-valued operator F.

Now assume that $\{x_n\}$ is a sequence in U converging to $x \in U$. There exists $\lambda > 0$ such that $F(x) = \lambda x^*$ for some $x^* \in T(x)$. From $\langle F(x), y - x\rangle = 1$ we deduce that $\lambda = 1/\langle x^*, x - y\rangle$. Since T is lsc, there exist $x_n^* \in T(x_n)$ such $\lim x_n^* = x^*$ in the weak* topology. There exist $\lambda_n > 0$ such that $F(x_n) = \lambda_n x_n^*$. Since $\langle \lambda_n x_n^*, x_n - y\rangle = 1$ by definition of $F(x_n)$, and $\lim \langle x_n^*, x_n - y\rangle = \langle x^*, x - y\rangle = 1/\lambda$, we deduce that $\lim \lambda_n = \lambda$. Consequently, $\lim F(x_n) = F(x)$ in the weak* topology, and F is norm-to-weak*continuous.

For part (iii) we have only to consider the case $x_0 \in Z_T$. Let U be a closed ball with center x_0 such that $U \subseteq D(T)$ and T is lower semicontinuous on U. Define $F : U \to X^*$ as follows: if $x \in Z_T$ then $F(x) = 0$. If $x \notin Z_T$, then choose $x^* \in T(x)$ and set $F(x) = \rho(x)\frac{x^*}{\|x^*\|}$ where $\rho(x)$ is the distance of x from the closed convex set $Z_T \cap U$. In view of (i), it is clear that F is equivalent to T. Also, F is obviously continuous at every point of $Z_T \cap U$. Continuity at points $x \in U \backslash Z_T$ can be proved as in case (ii): if the sequence $\{x_n\}$ in U converges to x and $F(x) = \rho(x)\frac{x^*}{\|x^*\|}$ with $x^* \in T(x)$, then by lower semicontinuity, there exist $x_n^* \in T(x_n)$ such that the sequence $\{x_n\}$ converges to x^*. By the finite dimensionality of the space and the continuity of ρ, we deduce that $\{F(x_n)\}$ converges to $F(x)$, hence F is continuous at x. □

Note that lower semicontinuity that we assumed in Proposition 5.7, even in the norm×norm topology in $X \times X^*$, is a weaker assumption than the Aubin property assumed in P7; see [21].

6 Paramonotone and Pseudomonotone$_*$ Operators

Many of the algorithms used to solve problems involving monotone operators make use of assumptions stronger than monotonicity. One of those assumptions is strict monotonicity: An operator $T : X \rightrightarrows X^*$ is called strictly monotone iff for every (x, x^*), $(y, y^*) \in \mathrm{gr}(T)$ with $x \neq y$,

$$\langle y^* - x^*, y - x \rangle > 0.$$

Likewise, the operator T is called strictly pseudomonotone iff for every (x, x^*), $(y, y^*) \in \mathrm{gr}(T)$ with $x \neq y$, the following implication holds:

$$\langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle > 0.$$

Strict monotonicity and pseudomonotonicity have some important consequences; for instance, they imply that VI has at most one solution. In search for less restrictive assumptions, Bruck [22] proposed the following property.

Definition 6.1 *A monotone operator T is called paramonotone iff for every (x, x^*), $(y, y^*) \in \mathrm{gr}(T)$,*

$$\langle y^* - x^*, y - x \rangle = 0 \Rightarrow x^* \in T(y) \textit{ and } y^* \in T(x).$$

Obviously, strictly monotone maps are paramonotone. It can be shown [22] that the subdifferential of any proper, lsc convex function is paramonotone, so paramonotonicity is a significantly less restrictive assumption than strict monotonicity. In fact, the following proposition holds:

Proposition 6.1 *If $T : X \rightrightarrows X^*$ is cyclically monotone and maximal monotone, then it is paramonotone. Consequently, the subdifferential of a proper, lsc convex function is paramonotone.*

The reason of the usefulness of paramonotone operators is that they possess the "cutting plane property" (CCP) which is stated as follows, where K is a subset of X:

$$\left.\begin{array}{c} x \in S(T, K) \\ z \in K \\ \langle z^*, x - z \rangle \geq 0 \text{ for some } z^* \in T(z) \end{array}\right\} \Rightarrow z \in S(T, K). \qquad \text{(CPP)}$$

CCP is useful in algorithms to solve VI for the following reason. Assume that VI has solutions. If at the nth iteration of an algorithm we find a point x_n that is not a solution of the Stampacchia variational inequality, then CCP implies that all solutions are contained in the intersection of K with the open halfspace $\{x \in X : \langle x_n^*, x - x_n \rangle < 0\}$, where x_n^* is any element of x_n.

A first attempt to find a notion similar to paramonotonicity, but related to pseudomonotone operators, was made by Crouzeix, Marcotte and Zhu [23], who introduced the following definition: a multivalued operator $T : X \rightrightarrows X^*$ is called[5] s-pseudomonotone$_*$ iff it is pseudomonotone, and for every (x, x^*), $(y, y^*) \in \mathrm{gr}(T)$ the following implication holds:

$$\langle x^*, y - x\rangle = \langle y^*, y - x\rangle = 0 \Rightarrow \exists k > 0 : ky^* \in T(x) .$$

It can be easily shown that s-pseudomonotone$_*$ operators satisfy CCP; also, every paramonotone operator is s-pseudomonotone$_*$. Finally, if f is a differentiable pseudoconvex function, then its gradient is s-pseudomonotone$_*$ [23]. However, until very recently, it was not known whether this can be generalized to nonsmooth pseudoconvex functions. An answer was given by Castellani and Giuli, who have shown by a counterexample that the Clarke subdifferential of a locally Lipschitz pseudoconvex function is not necessarily s-pseudomonotone$_*$. So this notion is not a good candidate to replace paramonotonicity for multivalued operators [24]. A better candidate is given through the use of equivalence of pseudomonotone operators.

Definition 6.2 *A pseudomonotone operator T is called pseudomonotone$_*$* [25] *iff it is pseudomonotone, and for every (x, x^*), $(y, y^*) \in \mathrm{gr}(T)$, the following implication holds:*

$$\langle x^*, x - y\rangle = \langle y^*, x - y\rangle = 0 \Rightarrow x^* \in \widehat{T}(y) \text{ and } y^* \in \widehat{T}(x) .$$

Pseudomonotone$_*$ operators have many nice properties. First, all paramonotone and all s-pseudomonotone$_*$ operators are pseudomonotone$_*$. In fact, a single-valued operator is s-pseudomonotone$_*$ if and only if it is pseudomonotone$_*$ [25]. This is no longer true if we consider multivalued operators [24]. In addition, the following proposition holds, which is a version of Proposition 6.1 adapted to pseudomonotone$_*$ operators.

Proposition 6.2 *If T is a D-maximal pseudomonotone, cyclically pseudomonotone operator with convex domain, then T is pseudomonotone$_*$. Consequently, the subdifferential of a locally Lipschitz pseudoconvex function is pseudomonotone$_*$.*

It is easy to see that if T is pseudomonotone$_*$ and $f : X \to]0, +\infty[$ is any function, then the operator T_1 defined by $T_1(\cdot) := f(\cdot)T(\cdot)$ is pseudomonotone$_*$. This can be generalized as follows [25]:

Proposition 6.3 *If $T : X \rightrightarrows X^*$ is pseudomonotone$_*$ and $S \sim T$, then S is pseudomonotone$_*$.*

It is not hard to check that pseudomonotone$_*$ operators have the cutting plane property. It is interesting that under the assumptions commonly used in variational inequalities, they are exactly the operators having the cutting plane property. Given $K \subseteq X$ and an operator T, let us call T "pseudomonotone on K" (resp., "pseudomonotone$_*$ on K), iff its restriction to K is pseudomonotone (resp., pseudomonotone$_*$). The proof of the following two results can be found in [25].

[5] In order to avoid confusion with the notion of pseudomonotone$_*$ operator defined below, we follow [24] and use "s-pseudomonotone$_*$" for the notion introduced in [23].

Theorem 6.1 *Let $T : X \rightrightarrows X^*$ be pseudomonotone on a convex set K.*

(i) If T is pseudomonotone$_*$, then CCP holds on every subset of K.

(ii) Conversely, if T has convex, w^*-compact values and has the CCP on every convex, compact subset of K, then T is pseudomonotone$_*$ on $\operatorname{int} K$.

In case T is single-valued, we can even drop the assumption of pseudomonotonicity:

Corollary 6.1 *Let $T : K \to X^*$ be hemicontinuous (i.e., weak*-continuous along straight lines). If T has the CCP on every convex compact subset of K, then it is pseudomonotone on K, and pseudomonotone$_*$ on* $\operatorname{int} K$.

As expected, algorithms making use of CCP can be successfully applied to pseudomonotone$_*$ operators. Three such examples can be found in [19, 25]. Another application is the following. Consider the minimization problem

$$\min\{f(x) : x \in C\} \tag{8}$$

where $C \subseteq X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper. The following well-known fact is the so-called minimum principle: if C is convex and f is convex and continuous at $\bar{x} \in C$, then $\bar{x}$ is a solution of (8) if and only if $\partial f(\bar{x}) \cap -N(\bar{x}, C) \neq \emptyset$. In relation to the minimum principle, Burke and Ferris [26] obtained the following result. Assume that C is closed and convex and f is continuous and convex. For any solution $\bar{x}$ of (8), the set S of all solutions is given by

$$S = \{x \in C : \partial f(x) \cap -N(x, C) = \partial f(\bar{x}) \cap -N(\bar{x}, C)\}. \tag{9}$$

Whenever f is locally Lipschitz and pseudoconvex, the minimum principle also holds in the following form: $\bar{x} \in C$ is a solution of (8) if and only if $\partial^{\uparrow} f(\bar{x}) \cap -N(\bar{x}, C) \neq \emptyset$. Very recently, by using Proposition 6.2, Castellani and Giuli [24] obtained the following result.

Proposition 6.4 *Let C be closed and convex, f a locally Lipschitz pseudoconvex function, and S the set of solutions of problem (8). If $\bar{x} \in S$, then*

$$S = \{x \in C : \widehat{\partial^{\uparrow}} f(x) \cap -N(x, C) = \widehat{\partial^{\uparrow}} f(\bar{x}) \cap -N(\bar{x}, C)\}.$$

Comparing with (9), we note that ∂f has to be replaced not just by $\partial^{\uparrow} f$, but with the maximum element of its equivalence class.

7 Pseudomonotone Operators and Mathematical Economics

Pseudomonotonicity appears in consumer theory of mathematical economics. Suppose that in an economy there are n different commodities. In this case, a *commodity bundle* is an element of $\mathbb{R}^n_+$ or, more generally, of a subset K of $\mathbb{R}^n_+$. It is supposed that for each consumer, a preference relation $\succeq$ is defined on K,

$x \succeq y$ meaning that the consumer likes x at least as much as y. In many cases, the preference relation is supposed to be defined by the so-called *utility function*, i.e., a function $u : K \to \mathbb{R}$ such that $x \succeq y$ if and only if $u(x) \geq u(y)$. Prices of the n commodities are represented by an element $p \in \mathbb{R}^n_{++} = (\mathbb{R}_{++})^n$. Given a number $w > 0$ representing the budget of the consumer, for each price vector p the consumer chooses a commodity bundle maximizing the utility, among all bundles x whose value $\langle p, x\rangle$ does not exceed w. That is, one has to solve the following problem:

$$\text{maximize} \quad u(x) \quad \text{s.t. } x \in K, \quad \langle p, x\rangle \leq w.$$

It is clear that the solutions of the above problem do not change if one replaces w by 1 and p by p/w. For this reason, w is usually taken equal to 1. Then one defines the *budget set* $B(p)$ by

$$B(p) := \{x \in K : \langle p, x\rangle \leq 1\}.$$

The *demand correspondence* $X : \mathbb{R}^n_{++} \rightrightarrows K$ is defined by

$$X(p) := \{x \in B(p) : u(x) \geq u(y),\ \forall y \in B(p)\}. \tag{10}$$

The utility function is a mathematically convenient, but artificial way to express consumer's preferences. The demand correspondence is supposed to have a more objective character than the utility function since it can actually be observed, and can be defined in an obvious way by using only the preference relation $\succeq$. The *revealed preference problem* in consumer theory is the following: given a demand correspondence X, does there exist a utility function u such that X is defined by (10), and, if the answer is positive, how can one define u using X? The answer should be related of course to some assumptions on the demand correspondence, that should be the result of economic considerations. Note that if u is a utility function and $h : u(X) \to \mathbb{R}$ is strictly increasing, then $h \circ u$ is a utility function corresponding to the same preference relation and producing the same demand correspondence. Given this non-uniqueness of u, in some cases one chooses a utility function satisfying a normalization condition, such as: $u(te) = t$ for all $t > 0$, where $e = (1, 1, \ldots 1)$.

The utility function u or, more generally, the preference relation $\succeq$ is assumed to satisfy some assumptions. For instance, it is assumed that u is continuous and strictly increasing along half-lines starting at the origin, i.e., $u(x) < u(tx)$ for all $t > 1$. This simply expresses that fact that all commodities are desirable by the consumer. In this case, whenever for some $x \in K$ one has $\langle p, x\rangle < 1$, it follows easily that $x \in B(p)\backslash X(p)$. Thus, for any $x \in X(p)$ one has $\langle p, x\rangle = 1$.

Now assume that $p_1, p_2, \ldots p_n$ is a finite sequence of prices and $x_i \in X(p_i)$ for $i = 1, 2, \ldots n$. Then $\langle p_i, x_i\rangle = 1$ for all i. Assume further that $\langle p_i, x_{i+1} - x_i\rangle \leq 0$ for all $i = 1, 2, \ldots n-1$. Then $\langle p_i, x_{i+1}\rangle \leq 1$ thus $x_{i+1} \in B(p_i)$. By definition of the correspondence X, this implies that $u(x_{i+1}) \leq u(x_i)$ for all $i = 1, 2, \ldots n-1$. Hence, $u(x_n) \leq u(x_1)$. It follows that necessarily $\langle p_n, x_1\rangle \geq 1$, otherwise $x_1 \in B(p_n)\backslash X(p_n)$ so that $u(x_1) < u(x_n)$, a contradiction. Consequently,

$\langle p_n, x_1 - x_n \rangle \geq 0$. The preceding argument shows that the demand correspondence must satisfy the so-called "Generalized Axiom of Revealed Preference" GARP [27]:

$$\left.\begin{array}{c} x_i \in X(p_i), \quad \forall i = 1, 2, \dots n \\ \langle p_i, x_{i+1} - x_i \rangle \leq 0, \quad \forall i = 1, 2, \dots n-1 \end{array}\right\} \Rightarrow \langle p_n, x_1 - x_n \rangle \geq 0. \qquad \text{(GARP)}$$

Denoting by X^{-1} the inverse of the operator X, GARP says simply that the operator $-X^{-1}$ is cyclically pseudomonotone.

The following property was remarked in [28].

Proposition 7.1 *Let Y be a Banach space and $T : Y \rightrightarrows Y^*$ be an operator such that $\langle x^*, x \rangle = 1$ for all $(x, x^*) \in \mathrm{gr}(T)$. Then T is cyclically pseudomonotone if and only if T^{-1} is cyclically pseudomonotone.*

Proof. Assume that T^{-1} is cyclically pseudomonotone. Consider any finite sequence $x_1, x_2, \dots x_n$ in Y and any $x_i^* \in T(x_i)$, $i = 1, 2, \dots n$. Now consider the finite sequence $x_n^*, x_{n-1}^*, \dots x_1^*$ (in this order). Since $x_i \in T^{-1}(x_i^*)$, cyclic pseudomonotonicity of T^{-1} says that the following implication is true:

$$\begin{aligned} &\left\langle x_{n-1}^* - x_n^*, x_n \right\rangle \geq 0, \left\langle x_{n-2}^* - x_{n-1}^*, x_{n-1} \right\rangle \geq 0, \dots \langle x_1^* - x_2^*, x_2 \rangle \geq 0 \\ &\qquad \Rightarrow \langle x_n^* - x_1^*, x_1 \rangle \leq 0. \end{aligned}$$

Since $\langle x_i^*, x_i \rangle = 1$, this implies

$$\left\langle x_{n-1}^*, x_n \right\rangle \geq 1, \left\langle x_{n-2}^*, x_{n-1} \right\rangle \geq 1, \dots \langle x_1^*, x_2 \rangle \geq 1 \Rightarrow \langle x_n^*, x_1 \rangle \leq 1$$

or, if we write the left-hand side in reverse order,

$$\begin{aligned} &\langle x_1^*, x_2 - x_1 \rangle \geq 0, \langle x_2^*, x_3 - x_2 \rangle \geq 0, \dots \left\langle x_{n-1}^*, x_n - x_{n-1} \right\rangle \geq 0 \\ &\qquad \Rightarrow \langle x_n^*, x_1 - x_n \rangle \leq 0. \end{aligned}$$

The last implication expresses the cyclic pseudomonotonicity of T.

The converse follows from the fact that T is the inverse of T^{-1}. $\square$

According to the proposition, GARP says also that $-X$ is cyclically pseudomonotone.

Very recently, Crouzeix, Keraghel and Rahmani [29] considered the case $K = \mathbb{R}^n_{++}$ and showed the following result. For the sake of this theorem, given a function $u : \mathbb{R}^n_{++} \to \mathbb{R}$, let us denote by X_u the demand correspondence defined by u. We call u increasing iff $u(x) \leq u(x + y)$ for all $x, y \in \mathbb{R}^n_{++}$.

Theorem 7.1 *Assume that $X : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}^n_{++}$ has nonempty values, satisfies GARP and is such that $\langle p, x \rangle = 1$ for all $x \in X(p)$, $p \in \mathbb{R}^n_{++}$. Then there exist two increasing, quasiconcave functions u_- and u_+ defined on $\mathbb{R}^n_{++}$, satisfying the normalization condition*

$$u_+(te) = u_-(te) = t, \ \forall t > 0$$

and such that, for all $x \in X(p)$, $p \in \mathbb{R}^n_{++}$, and all $y \in B(p)$,

$$u_+(x) \geq u_+(y) \text{ and } u_-(x) \geq u(y)$$

that is, $X(p) \subseteq X_{u_+}(p)$ and $X(p) \subseteq X_{u_-}(p)$ for all $p \in \mathbb{R}^n_{++}$.

If further $u : \mathbb{R}^n_{++} \to \mathbb{R}$ is any increasing, quasiconcave function satisfying $u(te) = t$, $\forall t > 0$ and such that $X(p) \subseteq X_u(p)$ for all $p \in \mathbb{R}^n_{++}$, then $u_-(x) \leq u(x) \leq u_+(x)$, for all $x \in \mathbb{R}^n_{++}$.

The above result is reminiscent of the result mentioned in Section 2, that the graph of any cyclically monotone operator is included in the graph of the subdifferential of a proper, lsc convex function.

8 Open questions

The theory of pseudomonotone operators is rich, but there are still many important open problems. We mention a few of them here.

1. Theorem 3.2 gives a characterization of single-valued, differentiable pseudomonotone operators. On the other hand, it is known that a single-valued differentiable operator T defined on a simply connected open subset of $\mathbb{R}^n$ is cyclically monotone if and only if T' is symmetric and positive semidefinite. Is there a characterization for single-valued, differentiable cyclically pseudomonotone operators?

2. Let $X : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}^n_{++}$ be an operator satisfying GARP and such that $\langle p, x\rangle = 1$ for all $x \in X(p)$, $p \in \mathbb{R}^n_{++}$. Which economically justified assumptions would guarantee that there exists a function $u : \mathbb{R}^n_{++} \to \mathbb{R}$ such that $X = X_u$ for all $p \in \mathbb{R}^n_{++}$, rather than $X(p) \subseteq X_u(p)$?

3. Under what conditions is a pseudomonotone operator equivalent to a monotone one?

4. Quasimonotone operators are in some sense more attractive than pseudomonotone ones. Besides having applications to economics, they are related to quasiconvex functions in a very simple way: a lsc function f is quasiconvex if and only if $\partial^{\uparrow} f$ is quasimonotone [5]. Of course, quasiconvex functions is a wider and more important class of functions than pseudoconvex ones. However, dealing with quasimonotone operators is often more tricky and requires more sophisticated arguments. Many of the advances for pseudomonotone operators in variational inequalities, have been also achieved, with some additional effort, for quasimonotone ones. However, there is no theory concerning the maximality of quasimonotone operators. Can one invent a suitable definition so that we have similar theoretical results and applications as for pseudomonotone operators?

5. The notion of D-maximal monotone operator seems to be very fruitful and fits very well with other notions such as pseudomonotone$_*$ operators. Many properties similar to those of monotone operators have been shown. However, there is a lot to be done: how to define maximal pseudomonotone operators without reference to the domain D? Can we have a theorem similar to Corollary 5.2 for the maximality of subdifferentials of lsc (rather than locally Lipschitz) pseudoconvex functions? How to extend Theorem 5.1 to infinite dimensional spaces?

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