Math. Program., Ser. B manuscript No.
(will be inserted by the editor)

Representative functions of maximally monotone operators and bifunctions

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Received: date / Accepted: date

Abstract The aim of this paper is to show that every representative function of a maximally monotone operator is the Fitzpatrick transform of a bifunction corresponding to the operator. In fact, for each representative function φ of the operator, there is a family of equivalent saddle functions (i.e., bifunctions which are concave in the first and convex in the second argument) each of which has φ as Fitzpatrick transform. In the special case where φ is the Fitzpatrick function of the operator, the family of equivalent saddle functions is explicitly constructed. In this way we exhibit the relation between the recent theory of representative functions, and the much older theory of saddle functions initiated by Rockafellar.

Keywords Maximal monotonicity · Fitzpatrick function · representative function · Fitzpatrick transform

Mathematics Subject Classification (2000) 47H05 · 47H04 · 49J53 · 90C33

Part of this work was done when the second author was visiting the Università Cattolica del Sacro Cuore, and the Università degli Studi di Milano-Bicocca, Italy. The author wishes to thank the Universities for their hospitality. The second author was supported by the startup research grant no. SR141001 of the King Fahd University of Petroleum and Minerals.

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1 Introduction

Given a maximally monotone operator T in a Banach space X, a class $\mathscr{H}(T)$ of convex, lower semicontinuous functions on the product space $X \times X^*$ was introduced by Fitzpatrick [6], that represent T in the following sense: each function $\varphi \in \mathscr{H}(T)$ determines exactly the graph of T as the set of coincidence of φ with the usual duality product. The class $\mathscr{H}(T)$ has a minimum element, the so-called Fitzpatrick function. The theory of representative functions has proven to be very fruitful, and has lead to major advances in the theory of maximally monotone operators; besides providing simpler proofs for known results such as the Debrunner-Flor theorem and the local boundedness in the interior of the domain [3], important new results have been found concerning the maximality of the sum of two maximally monotone operators in generally nonreflexive Banach spaces [12].

On the other hand, to every maximally monotone operator corresponds a class of bifunctions defined on the product $X \times X$. It had been shown that bifunctions, apart from being an interesting object of study in themseleves, especially in relation with equilibrium problems, are also useful for the study of maximally monotone operators. Actually, to every monotone operator corresponds a class of bifunctions such that, in some sense, the operator is the subdifferential of the bifunctions (see [9] for details). To each such bifunction, one defines its Fitzpatrick transform [4]. It has been shown that the Fitzpatrick transform of every bifunction corresponding to a maximally monotone operator, is a representative function of the operator [2]. One of the aims of the present paper, it to answer the following question: Does every representative function of a maximally monotone operator arise in this way? In other words, given a representative function, does there exist a bifunction corresponding to the operator, such that its Fitzpatrick transform is the given representative function? As we will see, the answer is yes, and in fact one may find all such bifunctions. In addition, these bifunctions may be chosen to be “saddle functions”, i.e., concave in the first variable and convex in the second one. Our results establish a close connection between the recent theory of representative functions and the much older theory of saddle functions by Rockafellar [18, 19], Krauss [10, 11] etc. In fact, some of our results are not really new; for example, Proposition 7 should be compared with Corollary 34.2.2 of [18]. What is new is their connection with the theory of maximally monotone operators and representative functions.

2 Preliminaries

Let X be a real Banach space and X^* its topological dual. Denote by π the duality product $\pi(x,x^*) = \langle x^*,x\rangle$. We will use the weak* topology in X^*, so its dual with respect to this topology is X. The space $X \times X^*$ is endowed with the product topology, so its dual is $X^* \times X$ with the canonical duality pairing defined by

$$\langle (x^*,x),(y,y^*)\rangle = \langle x^*,y\rangle + \langle y^*,x\rangle.$$

Given a subset K of X we will denote by $\mathrm{co}K$ and $\overline{\mathrm{co}}K$ its convex hull and closed convex hull, respectively; moreover, we will denote by δ_K the indicator function of

K, i.e.,

$$\delta_K(x) := \begin{cases} +\infty & \text{if } x \notin K, \\ 0 & \text{if } x \in K. \end{cases}$$

In the following we will denote by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty, +\infty\}$.

2.1 Some elements of convex analysis

In the sequel we recall some definitions according to [18]; it should be noted that some of the defined notions (such as closedness) differ from those found in other sources. Given a function $f : X \to \overline{\mathbb{R}}$, its domain and epigraph are, respectively, the sets $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ and $\operatorname{epi} f = \{(x,\mu) \in X \times \mathbb{R} : f(x) \leq \mu\}$. The function f is called convex if $\operatorname{epi} f$ is convex. The convex hull $\operatorname{co} f$ of a function f is the function which is the greatest convex minorant of f. Equivalently,

$$\begin{aligned} \operatorname{co} f(x) &= \inf\{\mu : (x,\mu) \in \operatorname{co}(\operatorname{epi} f)\} \\ &= \inf\{\sum_{i=1}^m \lambda_i f(x_i) : \sum_{i=1}^m \lambda_i x_i = x,\ x_i \in \operatorname{dom} f,\ \sum_{i=1}^m \lambda_i = 1,\ \lambda_i \geq 0\}. \end{aligned}$$

If f is convex, its *closure* $\overline{f}$ is defined as the pointwise supremum of all continuous affine functions majorized by f:

$$\overline{f} = \sup\{h : h \text{ is continuous affine}, h \leq f\}.$$

If f is convex and never takes the value $-\infty$, its closure $\overline{f}$ is the greatest lower semicontinuous (lsc) convex minorant of f; it is the function whose epigraph is the closure of $\operatorname{epi} f$. However, if f is convex and $f(x) = -\infty$ for some x, then $\overline{f} \equiv -\infty$. A convex function is said to be *closed* if $\overline{f} = f$. A convex function $f : X \to \overline{\mathbb{R}}$ is called *proper* if $f(x) > -\infty$, for every $x \in X$, and it is not identically equal to $+\infty$. For a proper convex function, closedness is the same as lower semicontinuity. For every function f, we denote by $\overline{\operatorname{co}} f$ the function $\overline{\operatorname{co} f}$.

The *convex conjugate* $f^* : X^* \to \overline{\mathbb{R}}$ of a function $f : X \to \overline{\mathbb{R}}$ is given by

$$f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}.$$

The function f^* is closed and convex, and it is proper if and only if f is proper. Moreover, $(\overline{\operatorname{co}} f)^* = \left(\overline{f}\right)^* = f^*$. In this paper, the convex conjugate of a function $g : X^* \to \overline{\mathbb{R}}$ will be meant to be defined in X rather than X^{**}. For every function f, $f^{**} = \overline{\operatorname{co}} f$.

For any function $f : X \to \overline{\mathbb{R}}$, the well-known Fenchel-Young inequality holds:

$$f^*(x^*) \geq \langle x^*, x\rangle - f(x) \text{ for all } x \in X, x^* \in X^*. \tag{1}$$

A function $f : X \to \overline{\mathbb{R}}$ is called concave if $-f$ is convex. Given a function f, its concave hull $\operatorname{cv} f$ is the function $\operatorname{cv} f = -\operatorname{co}(-f)$, i.e. the smallest concave majorant of f. Equivalently,

$$\operatorname{cv} f(x) = \sup\{\sum_{i=1}^m \lambda_i f(x_i) : \sum_{i=1}^m \lambda_i x_i = x,\ f(x_i) > -\infty,\ \sum_{i=1}^m \lambda_i = 1,\ \lambda_i \geq 0\}.$$

If f is concave, its closure is by definition the function $\overline{f} = -\overline{(-f)}$. In this case,

$$\overline{f} = \inf\{h : h \text{ is continuous affine}, h \geq f\}.$$

2.2 Monotone operators and representative functions

Given a multivalued operator $T : X \rightrightarrows X^*$, we recall that its domain and graph are, respectively, the sets $D(T) = \{x \in X : T(x) \neq \emptyset\}$ and $\operatorname{gph} T = \{(x,x^*) \in X \times X^* : x^* \in T(x)\}$.

In the sequel, we will assume that $D(T) \neq \emptyset$.

The multivalued operator T is called *monotone* if for any $x, y \in D(T)$ the inequality $\langle x^* - y^*, x - y\rangle \geq 0$ holds whenever $x^* \in T(x)$ and $y^* \in T(y)$. In particular, the monotone operator T is called *maximally monotone* if its graph is not properly included in the graph of any other monotone operator.

We recall that if T is maximally monotone and X is reflexive, then $\overline{D(T)}$ is convex, so $\overline{\operatorname{co}} D(T) = \overline{D(T)}$ [16].

Given a multivalued operator T, the class $\mathscr{H}(T)$ of *representative functions* of T is defined as the class of all closed and convex functions $\varphi : X \times X^* \to \overline{\mathbb{R}}$ such that:

$$\begin{cases} \varphi(x,x^*) \geq \langle x^*, x\rangle, \text{ for all } (x,x^*) \in X \times X^* \\ (x,x^*) \in \operatorname{gph} T \Rightarrow \varphi(x,x^*) = \langle x^*, x\rangle \end{cases}$$

Since we assume that $\operatorname{gph} T \neq \emptyset$, then any representative function is proper and thus, closedness is equivalent to lsc. With respect to each of its variables, φ might be improper, but it is still convex and closed.

To any operator $T : X \rightrightarrows X^*$, one associates its *Fitzpatrick function* [6] $\mathscr{F}_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} \mathscr{F}_T(x,x^*) &= \sup_{(y,y^*) \in \operatorname{gph} T} \left(\langle y^* - x^*, x - y\rangle + \langle x^*, x\rangle\right) \\ &= \sup_{(y,y^*) \in \operatorname{gph} T} \left(\langle x^*, y\rangle + \langle y^*, x - y\rangle\right). \end{aligned}$$

The Fitzpatrick function $\mathscr{F}_T$ is convex and lsc with respect to the pair (x,x^*). For any maximally monotone operator T, the function $\mathscr{F}_T$ belongs to $\mathscr{H}(T)$, and is in fact the smallest function of this family. In addition, for every $\varphi \in \mathscr{H}(T)$, the equality $\varphi(x,x^*) = \langle x^*, x\rangle$ characterizes the points in the graph of T.

On the other hand, the function $\sigma_T : X \times X^* \to \overline{\mathbb{R}}$ defined by

$$\sigma_T(x,x^*) := \overline{\operatorname{co}}\left(\pi + \delta_{\operatorname{gph} T}\right)(x,x^*)$$

is the greatest representative function in $\mathscr{H}(T)$, if T is maximally monotone [5].

The function σ_T is connected to the Fitzpatrick function via the following equalities:

$$\mathscr{F}_T(x,x^*) = \sigma_T^*(x^*,x), \qquad \mathscr{F}_T^*(x^*,x) = \sigma_T(x,x^*)$$

(see for instance [5, 13]).

In case of maximally monotone operators, the transpose of the conjugate of any representative function φ, i.e. the function $(\varphi^*)^t$ defined on $X \times X^*$ by $(\varphi^*)^t(x,x^*) = \varphi^*(x^*,x)$, where

$$\varphi^*(x^*,x) = \sup_{(y,y^*)\in X\times X^*} (\langle x^*,y\rangle + \langle y^*,x\rangle - \varphi(x,x^*)),$$

is also a representative function of T. Also, $\varphi(x,x^*) = \langle x^*,x\rangle$ implies $(x,x^*) \in \operatorname{gph} T$ [5].

Given a representative function φ, its domain is a subset of $X \times X^*$. We will denote by $P_1 \operatorname{dom}\varphi$ the projection of $\operatorname{dom}\varphi$ on X, i.e.,

$$P_1 \operatorname{dom}\varphi = \{x \in X : \exists x^* \in X^* \text{ such that } \varphi(x,x^*) < +\infty\}.$$

The following proposition is not new. We include a proof for completeness.

Proposition 1 *Let φ be a representative function of some operator T. Then*

a. $\operatorname{co} D(T) \subseteq P_1 \operatorname{dom}\varphi$.
b. If T is maximally monotone, then $P_1 \operatorname{dom}\varphi \subseteq \overline{\operatorname{co}} D(T)$. If in addition $\operatorname{int}\operatorname{co} D(T) \neq \emptyset$*, then* $\operatorname{int} D(T) = \operatorname{int} P_1 \operatorname{dom}\varphi$.

Proof a. Let $x \in D(T)$. Then there exists $x^* \in T(x)$, thus $\varphi(x,x^*) = \langle x^*,x\rangle \in \mathbb{R}$. Hence $D(T) \subseteq P_1 \operatorname{dom}\varphi$. Since $P_1 \operatorname{dom}\varphi$ is the projection of a convex set, it is convex, thus the inclusion $\operatorname{co} D(T) \subseteq P_1 \operatorname{dom}\varphi$ follows.

b. Let $x \in P_1 \operatorname{dom}\varphi$. Then there exists $x^* \in X^*$ such that $\varphi(x,x^*) \in \mathbb{R}$. Assume that $x \notin \overline{\operatorname{co}} D(T)$; then there exist $\varepsilon > 0$ and $v^* \in X^*$ such that $\langle v^*, x-y\rangle > \varepsilon$ for all $y \in D(T)$. We can choose v^* so that

$$\langle v^*, x-y\rangle \geq \varphi(x,x^*) - \langle x^*,x\rangle, \quad \forall y \in D(T).$$

Since the Fitzpatrick function $\mathscr{F}_T$ is the minimum element of the class of representative functions, for all $(y,y^*) \in \operatorname{gph} T$ we obtain

$$\langle x^* - y^*, y - x\rangle + \langle x^*,x\rangle \leq \mathscr{F}_T(x,x^*) \leq \varphi(x,x^*).$$

It follows that

$$\langle (x^* + v^*) - y^*, x - y\rangle \geq 0, \quad \forall (y,y^*) \in \operatorname{gph} T.$$

Since T is maximally monotone, $x^* + v^* \in T(x)$, contradicting $x \notin \overline{\operatorname{co}} D(T)$.

To show the equality of the interiors, we remark that $\operatorname{co} D(T) \subseteq P_1 \operatorname{dom}\varphi \subseteq \overline{\operatorname{co}} D(T)$ implies that $\operatorname{int} D(T) \subseteq \operatorname{int}\operatorname{co} D(T) \subseteq \operatorname{int} P_1 \operatorname{dom}\varphi \subseteq \operatorname{int}\overline{\operatorname{co}} D(T)$. If $\operatorname{int}\operatorname{co} D(T) \neq \emptyset$, then $\operatorname{int}\operatorname{co} D(T) = \operatorname{int}\overline{\operatorname{co}} D(T)$. In addition, it is known that $\operatorname{int} D(T) = \operatorname{int}\operatorname{co} D(T)$ [15], so we obtain $\operatorname{int} D(T) = \operatorname{int} P_1 \operatorname{dom}\varphi$. □

See also [21] for the inclusion $\mathrm{co}\,D(T) \subseteq P_1 \mathrm{dom}\mathscr{F}_T$, and [12] for the proof of the first part of b. Moreover, see [20, Theorem 2.2] for the equality $\mathrm{int}\,D(T) = \mathrm{int}\,P_1 \mathrm{dom}\mathscr{F}_T$.

Note that in general $\mathrm{co}\,D(T) \neq P_1 \mathrm{dom}\varphi \neq \overline{\mathrm{co}}D(T)$, as seen in the following example. Let $T : (0,1) \to \mathbb{R}$ be a continuous increasing function such that $T(x) = \frac{1}{1-x}$ near 1 and $T(x) = -\frac{1}{x^2}$ near 0. Then T is maximally monotone, and for every $x^* \geq 0$,

$$\mathscr{F}_T(1,x^*) = \sup_{y\in(0,1)} (T(y) + x^*y - yT(y)) \leq \sup_{y\in(0,1)} T(y)(1-y) + x^* < +\infty$$

while for every $x^* \in \mathbb{R}$,

$$\mathscr{F}_T(0,x^*) = \sup_{y\in(0,1)} (x^*y - yT(y)) = +\infty.$$

Hence $D(T) \neq P_1 \mathrm{dom}\mathscr{F}_T = (0,1] \neq \overline{\mathrm{co}}D(T)$.

2.3 Bifunctions and saddle functions

By the term *bifunction* we understand any function $F : X \times X \to \overline{\mathbb{R}}$. A bifunction F is said to be *normal* if there exists a nonempty set $C \subseteq X$ such that $F(x,y) = -\infty$ if and only if $x \notin C$. The set C will be called the *domain* of F and denoted by $D(F)$. In particular, if F is normal, then F is not identically $-\infty$.

The bifunction F is said to be monotone if

$$F(x,y) \leq -F(y,x)$$

for all $x,y \in X$. Every monotone bifunction satisfies the inequality $F(x,x) \leq 0$, for all $x \in X$.

Given a bifunction F, we define the operator $A^F : X \rightrightarrows X^*$ by

$$A^F(x) = \{x^* \in X^* : F(x,y) \geq \langle x^*, y-x\rangle,\ \forall y \in X\}.$$

Note that, if F is normal, then $D(A^F) \subseteq D(F)$, and

$$F(x,x) \geq 0 \quad \forall x \in D(A^F). \tag{2}$$

It is easy to check that the operator A^F is monotone whenever F is a monotone bifunction; moreover, $F(x,x) = 0$ for all $x \in D(A^F)$. The converse is not true: A^F may be monotone while F is not. See [8] for examples, and Proposition 4 below.

On the other hand, given an operator T one can define the bifunction $G_T : X \times X \to \overline{\mathbb{R}}$ by

$$G_T(x,y) = \sup_{x^*\in T(x)} \langle x^*, y-x\rangle. \tag{3}$$

The bifunction G_T is normal and $D(G_T) = D(T)$; furthermore $G_T(x,x) = 0$ for all $x \in D(T)$, and $G_T(x,\cdot)$ is closed and convex for all $x \in X$. If T is a monotone operator, then G_T is a monotone bifunction.

We can associate to each bifunction F its *Fitzpatrick transform*

$$\varphi_F(x,x^*) = \sup_{y\in X}(\langle x^*,y\rangle + F(y,x)) = (-F(\cdot,x))^*(x^*), \tag{4}$$

i.e., φ_F is the conjugate of $-F$ with respect to its first variable (see, for instance, [2] and [4]).

If $F(y,\cdot)$ is lsc and convex for all $y\in X$, then φ_F is also lsc and convex on $X\times X^*$. Moreover, if F is normal, then

$$\varphi_F(x,x^*) = \sup_{y\in X}(\langle x^*,y\rangle + F(y,x)) = \sup_{y\in D(F)}(\langle x^*,y\rangle + F(y,x));$$

this implies that $\varphi_F(x,x^*) > -\infty$ for all $(x,x^*)\in X\times X^*$, and φ_F is closed.

Note that, for any operator T, the following equality holds:

$$\begin{aligned}
\mathscr{F}_T(x,x^*) &= \sup_{(y,y^*)\in \operatorname{gph} T}(\langle x^*,y\rangle + \langle y^*,x-y\rangle) \\
&= \sup_{y\in X}\left(\langle x^*,y\rangle + \sup_{y^*\in T(y)}\langle y^*,x-y\rangle\right) \\
&= \sup_{y\in D(T)}(\langle x^*,y\rangle + G_T(y,x)) \\
&= \varphi_{G_T}(x,x^*).
\end{aligned} \tag{5}$$

Given a bifunction F, one can associate to F also the *upper Fitzpatrick transform* φ^F given by

$$\varphi^F(x,x^*) = \sup_{y\in X}(\langle x^*,y\rangle - F(x,y)) = F(x,\cdot)^*(x^*), \tag{6}$$

and the operator ${}^F\!A$, given by

$${}^F\!A(x) = \{x^*\in X^* : -F(y,x) \geq \langle x^*,y-x\rangle, \quad \forall y\in X\}$$

(see for instance [2], [4]).

A class of bifunctions widely used in mathematical literature is the class of *saddle* functions, i.e., bifunctions which are concave in the first argument, and convex in the second one (see, for instance, [18]). For these functions, let us recall some basic definitions.

One denotes by $\mathrm{cl}_2 F$ the bifunction obtained by closing $F(x,\cdot)$ as a convex function, for every $x\in X$; likewise, one denotes by $\mathrm{cl}_1 F$ the bifunction obtained by closing $F(\cdot,y)$ as a concave function, for every $y\in X$.

Two saddle functions F,H are called equivalent if $\mathrm{cl}_i F = \mathrm{cl}_i H$, $i=1,2$; in this case we write $F\sim H$. Clearly, $\sim$ is an equivalence relation. A saddle function F is called *closed* if $\mathrm{cl}_1 F \sim \mathrm{cl}_2 F \sim F$. It is called *lower closed* if $\mathrm{cl}_2\mathrm{cl}_1 F = F$, and *upper closed* if $\mathrm{cl}_1\mathrm{cl}_2 F = F$. It is easy to see that every lower closed and every upper closed saddle function is closed. Also, if $F\sim H$ and F is closed, then H is closed too.

Given a saddle function F, we define following [10]

$$\mathrm{dom}_1 F = \{x\in X : \mathrm{cl}_2 F(x,y) > -\infty, \quad \forall y\in X\},$$

and

$$\mathrm{dom}_2 F = \{y \in X : \mathrm{cl}_1 F(x,y) < +\infty, \quad \forall x \in X\}.$$

Note that, if F is a saddle function such that $\mathrm{cl}_2 F = F$ and F is not identically $-\infty$, then F is normal, and $\mathrm{dom}_1 F = D(F)$. Moreover, if F is a saddle function, such that $\mathrm{cl}_1 F = F$ and F is not identically $+\infty$, then $(x,y) \mapsto -F(y,x) = \hat{F}(x,y)$ is normal, and $\mathrm{dom}_2 F = D(\hat{F})$.

The next proposition shows that the quantities φ_F, φ^F, A^F and ${}^F A$ depend only on the equivalent class to which the saddle function F belongs.

Proposition 2 *Two saddle functions F and H are equivalent if and only if $\varphi_F = \varphi_H$ and $\varphi^F = \varphi^H$. In addition, if F and H are equivalent then $A^H = A^F$ and ${}^H A = {}^F A$.*

Proof If F is a saddle function, then

$$A^F = A^{\mathrm{cl}_2 F}, \quad \varphi_F = \varphi_{\mathrm{cl}_1 F}. \tag{7}$$

Here, the first equality stems from the definition of A^F and the closure of a convex function, while the second one is a consequence of relation (4) and the fact that $f^* = \left(\overline{f}\right)^*$ for every convex function f.

In a similar way as in (7), if F is a saddle function, then

$${}^F A = {}^{\mathrm{cl}_1 F} A, \quad \varphi^F = \varphi^{\mathrm{cl}_2 F}. \tag{8}$$

It follows from the above relations that whenever F and H are equivalent saddle functions, then $A^H = A^F$, ${}^H A = {}^F A$, $\varphi_H = \varphi_F$ and $\varphi^H = \varphi^F$.

Now assume that F and H are two saddle functions such that $\varphi_H = \varphi_F$ and $\varphi^H = \varphi^F$. From the first equality we deduce that

$$(-F(\cdot,x))^*(x^*) = (-H(\cdot,x))^*(x^*),$$

and, taking again the convex conjugate, we get that $\mathrm{cl}_1 F = \mathrm{cl}_1 H$. Moreover, from $\varphi^H = \varphi^F$, we get that

$$F(x,\cdot)^*(x^*) = H(x,\cdot)^*(x^*),$$

and, by taking the conjugates, we obtain that $\mathrm{cl}_2 F = \mathrm{cl}_2 H$. Thus, F and H are equivalent. □

3 The class of representative functions and saddle functions

Given a maximally monotone operator T, there is a whole family of representative functions $\mathscr{H}(T)$, one of which is its Fitzpatrick function.

In this section we will address the following question: given a maximally monotone operator T and one of its representative functions $\varphi \in \mathscr{H}(T)$, is it true that φ arises as the Fitzpatrick transform of a bifunction related to T? The answer is positive, and in addition the bifunction can be chosen to be a closed saddle function, as we will see in the sequel.

In the following proposition, we will show that, under suitable assumptions, both the Fitzpatrick transform and the upper Fitzpatrick transform of a bifunction F belong to $\mathscr{H}(T)$. We prove first a lemma:

Lemma 1 *Assume that T is a maximally monotone operator and $F : X \times X \to \overline{\mathbb{R}}$ is a bifunction such that $T(x) \subseteq A^F(x) \cap {}^F\!A(x)$ for all $x \in X$.*
(i) *If $F(x,\cdot)$ is lsc and convex for all $x \in X$, then $\varphi_F \in \mathscr{H}(T)$.*
(ii) *If $F(\cdot,y)$ is usc and concave for all $y \in X$, then $\varphi^F \in \mathscr{H}(T)$.*

Proof (i) Since $F(x,\cdot)$ is lsc and convex for all $x \in X$, φ_F is lsc and convex. Assume first that, for some $(x,x^*) \in X \times X^*$,

$$\varphi_F(x,x^*) \leq \langle x^*, x\rangle .$$

For every $(y,y^*) \in \mathrm{gph}(T)$, using successively that $T(x) \subseteq A^F(x)$ and the definition of φ_F,

$$\langle y^*, x-y\rangle + \langle x^*, y\rangle \leq \langle x^*, y\rangle + F(y,x) \leq \varphi_F(x,x^*) \leq \langle x^*, x\rangle . \tag{9}$$

Hence, $\langle y^* - x^*, y-x\rangle \geq 0$ for all $(y,y^*) \in \mathrm{gph}(T)$, so, by the maximality of T, $x^* \in T(x)$. Putting $y = x$ and $y^* = x^*$ in (9) we deduce that $\varphi_F(x,x^*) = \langle x^*, x\rangle$. It follows that $\varphi_F(x,x^*) < \langle x^*, x\rangle$ is not possible, hence $\varphi_F(x,x^*) \geq \langle x^*, x\rangle$ for all $(x,x^*) \in X \times X^*$.

To finish the proof that $\varphi_F \in \mathscr{H}(T)$, we need to show that $(x,x^*) \in \mathrm{gph}(T)$ implies $\varphi_F(x,x^*) = \langle x^*, x\rangle$. Indeed, from $T(x) \subseteq {}^F\!A(x)$ we deduce that, for all $y \in X$,

$$F(y,x) + \langle x^*, y\rangle \leq \langle x^*, x\rangle .$$

By taking the supremum for all $y \in X$ we get that $\varphi_F(x,x^*) \leq \langle x^*, x\rangle$. By the first part of the proof, $\varphi_F(x,x^*) = \langle x^*, x\rangle$.

(ii) We apply part (i) to the bifunction $\hat{F}(x,y) := -F(y,x)$. We note that $\hat{F}(x,\cdot)$ is lsc and convex for all $x \in X$, while $\varphi_{\hat{F}} = \varphi^F$, $A^{\hat{F}}(x) = {}^F\!A(x)$ and ${}^{\hat{F}}\!A(x) = A^F(x)$. We deduce that $\varphi^F = \varphi_{\hat{F}} \in \mathscr{H}(T)$. □

Note that in the above lemma we do not assume that F is monotone. In the special case of a monotone bifunction F, one has $A^F(x) \subseteq {}^F\!A(x)$, so the assumption $T(x) \subseteq A^F(x) \cap {}^F\!A(x)$ is equivalent to $T(x) = A^F(x)$ in view of the maximality of T.

Proposition 3 *Assume that T is a maximally monotone operator and $F : X \times X \to \overline{\mathbb{R}}$ is a closed saddle function. If $T(x) \subseteq A^F(x) \cap {}^F\!A(x)$ for all $x \in X$, then $\varphi_F \in \mathscr{H}(T)$ and $\varphi^F \in \mathscr{H}(T)$.*

Proof Since F is closed, $F \sim \mathrm{cl}_2 F$. By Proposition 2, $A^{\mathrm{cl}_2 F} = A^F$, ${}^{\mathrm{cl}_2 F}\!A = {}^F\!A$ and $\varphi_{\mathrm{cl}_2 F} = \varphi_F$. By applying part (i) of the Lemma 1 to $\mathrm{cl}_2 F$, we conclude that $\varphi_F = \varphi_{\mathrm{cl}_2 F} \in \mathscr{H}(T)$. Likewise, using $F \sim \mathrm{cl}_1 F$ and part (ii) of the lemma, we obtain $\varphi^F \in \mathscr{H}(T)$. □

In the main result of this section, we prove that all representative functions of T can be realized by taking the Fitzpatrick transform of suitable saddle functions.

In what follows, φ^* will be the convex conjugate (defined on $X^* \times X$) of φ with respect to the pair of variables (x,x^*), while expressions like $(\varphi^*(\cdot,x))^*(y)$ will mean the convex conjugate of φ^* (in X) with respect to the variable x^* only.

Given $\varphi \in \mathscr{H}(T)$, define the bifunction F by the formula

$$F(x,y) = \sup_{x^* \in X^*} \{\langle x^*, y\rangle - \varphi^*(x^*,x)\} = (\varphi^*(\cdot,x))^*(y) \tag{10}$$

By taking the second conjugate in (10) with respect to y we also find

$$\varphi^*(x^*,x) = (F(x,\cdot))^*(x^*) = \sup_{y\in X}\{\langle x^*,y\rangle - F(x,y)\}. \tag{11}$$

Theorem 1 *Let T be a maximally monotone operator and $\varphi \in \mathscr{H}(T)$. Then the bifunction F defined by the formula (10) has the following properties:*

(a) F is a saddle function such that $\mathrm{cl}_2 F = F$.
(b) F is normal, with $\mathrm{co}\,D(T) \subseteq D(F) \subseteq \overline{\mathrm{co}}D(T)$;
(c) $A^F = {}^F\!A = T$;
(d) $\varphi_F = \varphi$ and $\varphi^F = (\varphi^*)^t$.

Proof (a) For every $x \in X$, $F(x,\cdot)$ is the convex conjugate of a function, therefore it is closed and convex. In addition, for every $y \in X$, $F(x,y)$ is the supremum over x^* of a family of functions which are concave with respect to the pair (x,x^*); hence $F(\cdot,y)$ is concave (see Theorem 2.1.3(v) in [22]).

(b) Since $F(x,\cdot)$ is convex and closed, if $F(x,y_0) = -\infty$ for some (x,y_0), then $F(x,\cdot) = -\infty$; in particular, F is normal. In addition, it is evident that $x \in D(F)$ if and only if $\varphi^*(x^*,x) < +\infty$ for some $x^* \in X^*$, i.e., $D(F) = P_1\mathrm{dom}(\varphi^*)^t$. Since $(\varphi^*)^t$ is a representative function of T, the inclusions then follow from Proposition 1.

(c) Let us assume that $x^* \in T(x)$. Taking into account that $\varphi \in \mathscr{H}(T)$ entails that $(\varphi^*)^t \in \mathscr{H}(T)$ too, for all $y \in X$ we find

$$F(x,y) = \sup_{z^*\in X^*}\{\langle z^*,y\rangle - \varphi^*(z^*,x)\} \geq \langle x^*,y\rangle - \varphi^*(x^*,x) = \langle x^*,y-x\rangle\,;$$

hence, $T(x) \subseteq A^F(x)$.

Assume now that $x^* \in A^F(x)$. Then, taking into account (11), we find successively

$$\begin{aligned}\langle x^*,y-x\rangle \leq F(x,y),\ \forall y \in X &\Leftrightarrow \sup_{y\in X}\{\langle x^*,y\rangle - F(x,y)\} \leq \langle x^*,x\rangle \\ &\Leftrightarrow \varphi^*(x^*,x) \leq \langle x^*,x\rangle\,.\end{aligned}$$

Using again that $(\varphi^*)^t$ is a representative function, we find that $x^* \in T(x)$ so $A^F = T$.

Assume that $x^* \in {}^F\!A(x)$. This is equivalent to

$$\forall y \in X, \quad \langle x^*,y-x\rangle + F(y,x) \leq 0$$

i.e.,

$$\forall y \in X, \forall y^* \in X^*, \quad \langle x^*,y-x\rangle + \langle y^*,x\rangle - \varphi^*(y^*,y) \leq 0. \tag{12}$$

Since $(\varphi^*)^t$ is also a representative function, if we take $(y,y^*) \in \mathrm{gph}\,T$ then $\varphi^*(y^*,y) = \langle y^*,y\rangle$ so we deduce from (12) that

$$\forall (y,y^*) \in \mathrm{gph}\,T, \quad \langle y^*-x^*,y-x\rangle \geq 0.$$

From the maximality of T we deduce that $x^* \in T(x)$. Conversely, if $x^* \in T(x)$, then for every $(y,y^*) \in X \times X^*$ we find, using that $\mathscr{F}_T$ is the smallest representative function:

$$\varphi^*(y,y^*) \geq \mathscr{F}_T(y,y^*) \geq \langle x^*,y\rangle + \langle y^*,x\rangle - \langle x^*,x\rangle$$

so (12) holds. Hence $x^* \in {}^F\!A(x)$.

(d) Since φ is proper, lsc and convex, $\varphi^{**} = \varphi$. We have from (11), using also that $F(x,\cdot)$ is convex and closed,

$$
\begin{aligned}
\varphi(x,x^*) &= \sup_{(y^*,y)\in X^*\times X} (\langle y^*,x\rangle + \langle x^*,y\rangle - \varphi^*(y^*,y)) \\
&= \sup_{(y^*,y)\in X^*\times X} (\langle y^*,x\rangle + \langle x^*,y\rangle - (F(y,\cdot))^*(y^*)) \\
&= \sup_{y\in Y}(\langle x^*,y\rangle + \sup_{y^*\in X^*} (\langle y^*,x\rangle - (F(y,\cdot))^*(y^*)) \\
&= \sup_{y\in Y} (\langle x^*,y\rangle + (F(y,\cdot))^{**}(x)) \\
&= \sup_{y\in Y} (\langle x^*,y\rangle + F(y,x)) \\
&= \varphi_F(x,x^*).
\end{aligned}
$$

Finally, comparing (11) and (6) we get immediately $\varphi^F = (\varphi^*)^t$. □

Note that F is not monotone in general:

Proposition 4 *The bifunction F defined by (10) is monotone if and only if $\varphi(x,x^*) \leq \varphi^*(x^*,x)$, for every $x \in X$, $x^* \in X^*$.*

Proof In order to see when F is monotone, notice that the condition $F(y,x) \leq -F(x,y)$ is equivalent to

$$\langle y^*,x\rangle - \varphi^*(y^*,y) \leq -\langle x^*,y\rangle + \varphi^*(x^*,x), \quad \forall x,y \in X,\, x^*,y^* \in X^*,$$

or, alternatively,

$$\sup_{y\in X, y^*\in X^*} (\langle x^*,y\rangle + \langle y^*,x\rangle - \varphi^*(y^*,y)) \leq \varphi^*(x^*,x), \quad \forall x \in X, x^* \in X^*,$$

i.e.,

$$\varphi(x,x^*) \leq \varphi^*(x^*,x), \quad \forall x \in X, x^* \in X^*,$$

since $\varphi^{**} = \varphi$. □

The bifunction F defined by (10) is not the only saddle function that satisfies (c) and (d) of Theorem 1. According to Proposition 2, any saddle function equivalent to F also satisfies these conditions. An example of a saddle function equivalent to F is given by

$$\widetilde{F}(x,y) = -\sup_{y^*\in X^*} \{\langle y^*,x\rangle - \varphi(y,y^*)\} = -(\varphi(y,\cdot))^*(x) \tag{13}$$

Indeed, the next proposition holds:

Proposition 5 *The bifunction $\widetilde{F}$ is a saddle function and satisfies*

$$\widetilde{F} = \mathrm{cl}_1 F, \qquad F = \mathrm{cl}_2 \widetilde{F}. \tag{14}$$

Consequently, F is lower closed, $\widetilde{F}$ is upper closed, and $F \sim \widetilde{F}$. Finally,

$$F(x,y) \leq \widetilde{F}(x,y), \qquad \forall (x,y) \in X \times X.$$

Proof The proof that $\widetilde{F}$ is a saddle function is similar to the proof of the analogous assertion for F in Theorem $1(a)$. By Theorem 1 and relation (4),

$$\varphi(y,y^*) = \varphi_F(y,y^*) = (-F(\cdot,y))^*(y^*),$$

and therefore $\widetilde{F}$ is also given by the formula

$$-\widetilde{F}(x,y) = (-F(\cdot,y))^{**}(x) \tag{15}$$

i.e., $\widetilde{F} = \mathrm{cl}_1 F$. In addition, in view of (13),

$$\begin{aligned}\varphi^*(x^*,x) &= \sup_{(y,y^*)\in X\times X^*} \{\langle x^*,y\rangle + \langle y^*,x\rangle - \varphi(y,y^*)\} \\ &= \sup_{y\in X}\left\{\langle x^*,y\rangle + \sup_{y^*\in X^*}(\langle y^*,x\rangle - \varphi(y,y^*))\right\} \\ &= \sup_{y\in X}\left\{\langle x^*,y\rangle - \widetilde{F}(x,y)\right\} = \left(\widetilde{F}(x,\cdot)\right)^*(x^*).\end{aligned}$$

Therefore,

$$F(x,y) = (\varphi^*(\cdot,x))^*(y) = \left(\widetilde{F}(x,\cdot)\right)^{**}(y) = \mathrm{cl}_2\widetilde{F}(x,y).$$

The inequality $F \le \widetilde{F}$ follows from $F = \mathrm{cl}_2\widetilde{F}$.

The remaining assertions of the proposition are immediate consequences of equalities (14). $\square$

The next proposition summarizes some results about $\widetilde{F}$, similar to Theorem 1.

Proposition 6 *Let T be a maximally monotone operator, $\varphi \in \mathscr{H}(T)$ and $\widetilde{F}$ be defined by (13). Then:*

(a) *$\widetilde{F}$ is a saddle function such that* $\mathrm{cl}_1\widetilde{F} = \widetilde{F}$.
(b) *$-\widetilde{F}^t$ is normal, and* $\mathrm{co}D(T) \subseteq D(-\widetilde{F}^t) \subseteq \overline{\mathrm{co}}D(T)$, *where* $\widetilde{F}^t(x,y) = \widetilde{F}(y,x)$;
(c) $\varphi_{\widetilde{F}} = \varphi$ *and* $\varphi^{\widetilde{F}} = (\varphi^*)^t$;
(d) $T = A^{\widetilde{F}} = {}^{\widetilde{F}}\!A$.

Proof Parts (a), (c) and (d) follow from Propositions 2 and 5. The proof of (b) follows the same steps as the proof of Theorem $1(b)$. $\square$

In the next result, we prove that the set of all saddle functions that are equivalent to F is exactly the set of saddle functions pointwisely bounded from below and from above by $\widetilde{F}$, and F, respectively. Consequently, the bifunctions F and $\widetilde{F}$ play the role of maximal and minimal element in the class of saddle functions satisfying the equalities

$$\varphi_H = \varphi, \quad \varphi^H = (\varphi^*)^t. \tag{16}$$

Proposition 7 *Let H be a saddle function. Then H satisfies* (16) *if and only if* $F \le H \le \widetilde{F}$.

Proof It is easy to see that every saddle function H such that $F \leq H \leq \widetilde{F}$ is equivalent to F (because $\mathrm{cl}_1 F \leq \mathrm{cl}_1 H \leq \mathrm{cl}_1 \widetilde{F} = \mathrm{cl}_1 F$, and the same for cl_2), hence it satisfies (16). Conversely, if H satisfies (16), then $H \sim F$. From the equalities

$$\mathrm{cl}_1 H = \mathrm{cl}_1 F = \widetilde{F}, \quad \mathrm{cl}_2 H = \mathrm{cl}_2 F = F,$$

and since for every convex (concave) function the convex (concave) closure is smaller (greater) than the function, we get that $F \leq H \leq \widetilde{F}$. $\square$

We conclude by illustrating the particular case where $\varphi = \mathscr{F}_T$. We will construct the saddle functions F and $\tilde{F}$, defined by (10) and (13), and we will show how they are related to G_T.

In view of (4) and (5),

$$\mathscr{F}_T(x,x^*) = \varphi_{G_T}(x,x^*) = (-G_T(\cdot,x))^*(x^*). \tag{17}$$

Since the bifunction G_T is not saddle, in general, let us consider the bifunction $\hat{G}_T : X \times X \to \overline{\mathbb{R}}$ defined by $\hat{G}_T(\cdot,y) = \mathrm{cv}\, G_T(\cdot,y)$, for each $y \in X$ (see also [11,2]).

Since $G_T(x,y) > -\infty$ is equivalent to $x \in D(T)$, $\hat{G}_T$ is given by

$$\hat{G}_T(x,y) := \sup\{\sum_{i=1}^{k} \alpha_i G_T(x_i,y) : x = \sum_{i=1}^{k} \alpha_i x_i,\, x_i \in D(T), \sum_{i=1}^{k} \alpha_i = 1,\, \alpha_i \geq 0\}.$$

By construction, $\hat{G}_T(\cdot,y)$ is concave; also, $\hat{G}_T(x,\cdot)$ is convex and closed, as a supremum of convex and closed functions. Thus, $\hat{G}_T$ is a saddle function such that $\mathrm{cl}_2 \hat{G}_T = \hat{G}_T$.

Since T is monotone, we know that G_T is monotone, thus

$$G_T(x,y) \leq -G_T(y,x), \qquad \forall (x,y) \in X \times X.$$

If we take the convex hull with respect to y of both sides we find

$$G_T(x,y) \leq -\hat{G}_T(y,x), \qquad \forall (x,y) \in X \times X.$$

Now we take the concave hull with respect to x of both sides and we deduce

$$\hat{G}_T(x,y) \leq -\hat{G}_T(y,x), \qquad \forall (x,y) \in X \times X.$$

Consequently, $\hat{G}_T$ is monotone.

We have

$$\begin{aligned}
(\varphi_{G_T})^*(x^*,x) &= \sup_{(y,y^*) \in X \times X^*} \{\langle x^*,y\rangle + \langle y^*,x\rangle - \varphi_{G_T}(y,y^*)\} \\
&= \sup_{(y,y^*) \in X \times X^*} \{\langle x^*,y\rangle + \langle y^*,x\rangle - (-G_T(\cdot,y))^*(y^*)\} \\
&= \sup_{y \in X} \left\{\langle x^*,y\rangle + (-G_T(\cdot,y))^{**}(x)\right\} \\
&= \sup_{y \in X} \left\{\langle x^*,y\rangle - \mathrm{cl}_1 \hat{G}_T(x,y)\right\} \\
&= \left(\mathrm{cl}_1 \hat{G}_T(x,\cdot)\right)^*(x^*)
\end{aligned}$$

thus

$$F(x,y) = \left((\varphi_{G_T})^*(\cdot,x)\right)^*(y) = \left(\mathrm{cl}_1\hat{G}_T(x,\cdot)\right)^{**}(y) = \mathrm{cl}_2\mathrm{cl}_1\hat{G}_T(x,y).$$

That is, F is the "lower closure" of $\hat{G}_T$ [10]. Note that by Proposition 4, F is monotone, because $\mathscr{F}_T(x,x^*) \leq \sigma_T(x^*,x)$.

Since $\hat{G}_T$ is convex and closed in the second variable, $\mathrm{cl}_2\hat{G}_T = \hat{G}_T$. Using that for every saddle function H the saddle function $\mathrm{cl}_1\mathrm{cl}_2 H$ is upper closed [16,10] we find

$$\widetilde{F} = \mathrm{cl}_1 F = \mathrm{cl}_1\mathrm{cl}_2\mathrm{cl}_1\hat{G}_T = \mathrm{cl}_1\mathrm{cl}_2\mathrm{cl}_1\mathrm{cl}_2\hat{G}_T = \mathrm{cl}_1\mathrm{cl}_2\hat{G}_T = \mathrm{cl}_1\hat{G}_T.$$

Thus, $\widetilde{F}$ is the "upper closure" of $\hat{G}_T$.

Acknowledgements The authors would like to thank the referees for their suggestions that led to the improvement of the paper.

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