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RESEARCH ARTICLE

On the Fitzpatrick transform of a monotone bifunction

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A new definition of monotone bifunctions is given, which is a slight generalization of the original definition given by Blum and Oettli, but which is better suited for relating monotone bifunctions to monotone operators. In this new definition, the Fitzpatrick transform of a maximal monotone bifunction is introduced so as to correspond exactly to the Fitzpatrick function of a maximal monotone operator in case the bifunction is constructed starting from the operator. Whenever the monotone bifunction is lower semicontinuous and convex with respect to its second variable, the Fitzpatrick transform permits to obtain results on its maximal monotonicity.

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1. Introduction

Given a nonempty subset of a Banach space X, the term "monotone bifunction" on C is often used for functions $F : C \times C \to \mathbb{R}$ such that

$$F(x,y) + F(y,x) \le 0, \ \forall x, y \in C.$$

Starting from the paper by Blum and Oettli [5], monotone bifunctions were studied mainly in view of their application to equilibrium problems. Here, we will focus our interest on their relation to monotone operators, which can be done as follows. Given a multivalued monotone operator $T : X \rightrightarrows X^*$ with domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$, the bifunction G_T defined on $D(T) \times D(T)$ by

$$G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \tag{1}$$

is real-valued and monotone. On the other hand, given any monotone bifunction F, the operator defined by

$$A^F(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le F(x,y), \ \forall y \in C\}$$

whenever $x \in C$ while $A^F(x) = \emptyset$ for $x \notin C$, is monotone.

Also, in [5] a notion of maximality for bifunctions was introduced. A monotone bifunction $F : C \times C \to \mathbb{R}$ is called BO-maximal monotone if for all $x \in C$ and

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$x^* \in X^*$, the following implication holds:

$$F(y,x) + \langle x^*, y-x\rangle \le 0,\ \forall y \in C \Longrightarrow \langle x^*, y-x\rangle \le F(x,y),\ \forall y \in C.$$

The bifunction F is called maximal monotone if the operator A^F is maximal monotone. It can be shown that every maximal monotone bifunction is BO-maximal monotone [2]; the converse is not true in general [2], but it holds under some additional assumptions: For instance, if X is reflexive, F is BO-maximal monotone, C is closed and convex, $F(x,\cdot)$ is lsc and convex and $F(x,x)=0$ for every $x \in C$, then F is maximal monotone [1, 8].

A very powerful tool in the study of maximal monotone operators is the notion of Fitzpatrick function [11]. Given a monotone operator T with graph $\operatorname{gr} T = \{(x,x^*) \in X \times X^* : x^* \in T(x)\}$, its Fitzpatrick function $\mathcal{F}_T$ can be written as

$$\mathcal{F}_T(x,x^*) = \sup_{(y,y^*)\in \operatorname{gr} T} \left(\langle x^*, y\rangle + \langle y^*, x-y\rangle\right).$$

A lsc convex function φ on $X \times X^*$ is called a representative function of a monotone operator T if $\varphi(x,x^*) \ge \langle x^*, x\rangle$ for all $(x,x^*) \in X \times X^*$, and $\varphi(x,x^*) = \langle x^*, x\rangle$ for all $(x,x^*) \in \operatorname{gr} T$. It is known that the Fitzpatrick function of a maximal monotone operator T is a representative function of T. It has been shown recently that some important results on maximal monotone operator theory may be obtained by using methods of convex analysis on representative functions; see for instance [4, 6, 7, 13, 14] etc.

If we compare the definitions of $\mathcal{F}_T$ and G_T we obtain

$$\mathcal{F}_T(x,x^*) = \sup_{y\in D(T)} \left(\langle x^*, y\rangle + G_T(y,x)\right).$$

Note that $\mathcal{F}_T$ is defined for all $x \in X$ (although y needs only to be in $D(T)$), and that in fact formula (1) can be used to define G_T on all $X \times X$. Obviously, $G_T(x,y) = -\infty$ for $x \notin D(T)$. This motivates the definition of a Fitzpatrick transform for every monotone bifunction, but we need to have bifunctions defined on $X \times X$. In fact, such kind of functions were introduced in [8] for bifunctions $F : C \times C \to \mathbb{R}$, where it was shown that one can recover some nice results and find new ones by using tools of convex analysis. In this paper we will introduce the so-called "normal bifunctions" defined on $X \times X$ and taking on values from $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$; we will see that the new formulation includes the previous one and gives simpler, more appealing formulas. Note that in [3], one considers bifunctions $F : X \times X \to \overline{\mathbb{R}}$ and defines monotonicity with respect to a subset C by $F(x,y) \le -F(y,x)$, $x,y \in C$. However, all other definitions and all results in [3] actually concern the restriction of F on $C \times C$, where F is real.

We will use the weak* topology on X^*, so its dual with respect to this topology is X, and the dual of $X \times X^*$ is $X^* \times X$. Given a subset K of X, we will denote by $\operatorname{co} K$ and $\overline{\operatorname{co}} K$ its convex hull and weak*-closed convex hull, respectively.

Given a function $f : X \to \overline{\mathbb{R}}$, we will use its convex conjugate $f^* : X^* \to \overline{\mathbb{R}}$ and convex biconjugate $f^{**} : X \to \overline{\mathbb{R}}$ defined by

$$f^*(x) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\},$$
$$f^{**}(x) = \sup\{\langle x^*, x\rangle - f^*(x^*) : x^* \in X^*\}.$$

We refer the reader to the excellent treatise [16] for their properties.

2. BO-maximal monotone bifunctions

In what follows, X will be a Banach space.

Definition 2.1: A function $F : X \times X \to \overline{\mathbb{R}}$ is called normal bifunction if there exists a nonempty subset C of X such that

$$F(x, y) = -\infty \text{ iff } x \notin C.$$

C will be called the domain of F. In what follows, it will be denoted by $D(F)$.

According to Definition 2.1, $F : X \times X \to \overline{\mathbb{R}}$ is a normal bifunction if and only if we have that

$$\begin{aligned} &\{x \in X : \exists y \in X \text{ such that } F(x, y) > -\infty\} \\ &= \{x \in X : \forall y \in X,\ F(x, y) > -\infty\} \neq \emptyset. \end{aligned}$$

In this case C coincides with the sets from above. Note also that in this definition we do not impose the assumption $F(x, x) = 0$ for all $x \in D(F)$.

Definition 2.2: A normal bifunction $F : X \times X \to \overline{\mathbb{R}}$ is called monotone if

$$F(x, y) \leq -F(y, x)\,,\ \forall x, y \in X. \tag{2}$$

Remark 1: Let $F : X \times X \to \overline{\mathbb{R}}$ be a monotone bifunction. If x and y are both in $D(F)$, then $F(y, x) > -\infty$ and so $-F(y, x) < +\infty$, thus $-\infty < F(x, y) < +\infty$. In a similar manner we get $F(y, x) \in \mathbb{R}$. Hence we see that a normal bifunction is monotone if and only if

$$F(x, y) + F(y, x) \ \leq 0,\ \forall x, y \in D(F),$$

see also [3]. Therefore for all x in $D(F)$, we have $F(x, x) \leq 0$.

For any operator $T : X \rightrightarrows X^*$ one can define a normal bifunction G_T with domain $D(G_T) = D(T)$ by the formula

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle\,,\ \forall x, y \in X.$$

Then $G_T(x, x) = 0$ for all x in $D(T)$. Moreover $G_T(x, \cdot)$ is lsc and convex for all x in $D(T)$.

Let $F : X \times X \to \overline{\mathbb{R}}$ be a normal bifunction. Define the operator A^F by

$$A^F(x) = \{x^* : \langle x^*, y - x\rangle \leq F(x, y)\,,\ \forall y \in X\}\,. \tag{3}$$

One can easily check that $D(A^F) \subseteq D(F)$; also, whenever F is monotone, A^F is also monotone and one has $F(x, x) = 0$ for all $x \in D(A^F)$.

Remark 2: So far, papers in the literature consider a bifunction to be defined on $C \times C$, where C is a subset of X, and define A^F by requiring (3) to hold for $x, y \in C$. This is a particular case of what we are considering here. Indeed, for any $F : C \times C \to \mathbb{R}$ one can define a normal bifunction $\widetilde{F} : X \times X \to \overline{\mathbb{R}}$ which extends

F on the whole space, by setting

$$\widetilde{F}(x,y) = \begin{cases} F(x,y), \, x \in C \text{ and } y \in C \\ +\infty, \quad x \in C \text{ and } y \notin C \\ -\infty, \qquad\quad x \notin C. \end{cases}$$

Then $A^{\widetilde{F}}$ satisfies

$$A^{\widetilde{F}}(x) = \begin{cases} \{x^* : \langle x^*, y-x \rangle \le F(x,y), \ \forall y \in C\}, \, x \in C \\ \emptyset, \qquad\qquad\qquad\qquad\qquad\qquad x \notin C \end{cases}$$

i.e., it is the operator A^F considered in previous papers.

In the same spirit, we redefine the notion of BO-maximality.

Definition 2.3: A monotone bifunction F is called BO-maximal monotone if for all $(x, x^*) \in X \times X^*$,

$$F(y,x) + \langle x^*, y-x \rangle \le 0, \ \forall y \in X \Longrightarrow \langle x^*, y-x \rangle \le F(x,y), \ \forall y \in X. \quad (4)$$

A monotone bifunction F is called maximal monotone if A^F is maximal monotone.

Note that F is BO-maximal monotone if and only if

$$F(y,x) + \langle x^*, \ y-x \rangle \le 0, \ \forall y \in D(F) \Longrightarrow \langle x^*, y-x \rangle \le F(x,y), \ \forall y \in X. \quad (5)$$

It is easy to check that every maximal monotone bifunction is BO-maximal monotone.

Remark 3: The right-hand side of (4) says that $x^* \in A^F(x)$ and, consequently, $x \in D(A^F)$; thus, if F is BO-maximal monotone and $F(y,x) + \langle x^*, \ y-x \rangle \le 0$ holds for some $x \in X$ and for every $y \in D(F)$, then $x \in D(F)$ and $F(x,x) = 0$.

In view of Remark **2**, the definition of BO-maximal monotonicity considered in previous papers where F is defined on $C \times C$ and the right-hand side of (5) is required to hold only for $y \in C$, is again a particular case of the definition considered here.

Given an operator T, we denote by $\overline{\text{co}}T$ the operator whose value at each $x \in X$ is $\overline{\text{co}}(T(x))$. Then $G_T = G_{\overline{\text{co}}T}$.

Proposition 2.4: *Let $T : X \rightrightarrows X^*$ be an operator. Then $\overline{\text{co}}T$ is maximal monotone if and only if G_T is BO-maximal monotone.*

Proof: Let $\overline{\text{co}}T$ be maximal monotone. Since $G_T = G_{\overline{\text{co}}T}$ we may suppose without loss of generality that T is maximal monotone. Now assume that T is maximal monotone and for some $(x, x^*) \in X \times X^*$,

$$G_T(y,x) + \langle x^*, y-x \rangle \le 0, \ \forall y \in X.$$

Then

$$\sup_{y^* \in T(y)} \langle y^*, x-y \rangle + \langle x^*, y-x \rangle \le 0, \ \forall y \in D(T).$$

Thus for all $(y, y^*) \in \operatorname{gr} T$,

$$\langle y^* - x^*, y - x \rangle \geq 0.$$

Since T is maximal monotone, we deduce $(x, x^*) \in \operatorname{gr} T$. This implies that for every $y \in X$,

$$\langle x^*, y - x \rangle \leq \sup_{z^* \in T(x)} \langle z^*, y - x \rangle = G_T(x, y).$$

Thus G_T is BO-maximal.

Conversely, suppose that $G_T = G_{\overline{\text{co}}T}$ is BO-maximal monotone. Then, for all $x, y \in D(T), x^* \in \overline{\text{co}}T(x)$ and $y^* \in \overline{\text{co}}T(y)$,

$$\langle y^* - x^*, y - x \rangle \geq -(G_{\overline{\text{co}}T}(x, y) + G_{\overline{\text{co}}T}(y, x)) \geq 0.$$

It follows that $\overline{\text{co}}T$ is monotone. To show that it is maximal monotone, let $(x, x^*) \in X \times X^*$ be such that $\langle y^* - x^*, y - x \rangle \geq 0$ for all $(y, y^*) \in \operatorname{gr} \overline{\text{co}}T$. Then $\langle x^*, y - x \rangle + \langle y^*, x - y \rangle \leq 0$ for all $(y, y^*) \in \operatorname{gr} \overline{\text{co}}T$. By taking the supremum over $y^* \in T(y)$ we get $G_T(y, x) + \langle x^*, y - x \rangle \leq 0$ for all $y \in X$. Since $G_T = G_{\overline{\text{co}}T}$ is BO-maximal, we deduce

$$\langle x^*, y - x \rangle \leq G_T(x, y) = \sup_{z^* \in \overline{\text{co}}T(x)} \langle z^*, y - x \rangle, \ \forall y \in X \tag{6}$$

Since $\overline{\text{co}}T(x)$ is closed and convex, (6) together with the separation theorem imply that $x^* \in \overline{\text{co}}T(x)$. □

3. The Fitzpatrick transform of a monotone bifunction

The following is an adaptation of a corresponding definition of [8]:

Definition 3.1: Suppose that $F : X \times X \to \overline{\mathbb{R}}$ is a monotone bifunction. Define its Fitzpatrick transform $\varphi_F : X \times X^* \to \overline{\mathbb{R}}$ by

$$\varphi_F(x, x^*) = \sup_{y \in X} (\langle x^*, y \rangle + F(y, x)), \ \forall (x, x^*) \in X \times X^*.$$

Whenever $F(y, \cdot)$ is lsc and convex for all $y \in D(F)$, then φ_F is also lsc and convex.

For every BO-maximal monotone bifunction we have the following theorem, which is similar to a corresponding theorem for the Fitzpatrick function of a maximal monotone operator; in case $F(x, \cdot)$ is convex and lsc, the theorem says that φ_F is a representative function for the operator A^F.

Theorem 3.2: *Assume that F is a BO-maximal monotone bifunction. For each $(x, x^*) \in X \times X^*$ one has $\langle x^*, x \rangle \leq \varphi_F(x, x^*)$. Equality holds if and only if $x^* \in A^F(x)$.*

Proof: Suppose that for some $(x, x^*) \in X \times X^*$ one has

$$\varphi_F(x, x^*) \leq \langle x^*, x \rangle. \tag{7}$$

Then $\sup_{y\in X}\left(\langle x^*,y\rangle+F(y,x)\right)\le\langle x^*,x\rangle$, thus $F\left(y,x\right)+\langle x^*,y-x\rangle\le 0, \forall y\in X$. By assumption F is BO-maximal, therefore $\langle x^*,y-x\rangle\le F\left(x,y\right), \ \forall y\in X$. By Remark **3**, this implies that $x\in D(F)$ and $F\left(x,x\right)=0$, thus

$$\varphi_F(x,x^*)=\sup_{y\in X}\left(\langle x^*,y\rangle+F(y,x)\right)\ge\langle x^*,x\rangle+F(x,x)=\langle x^*,x\rangle\,. \tag{8}$$

Now from (7) and (8) we get $\varphi_F(x,x^*)=\langle x^*,x\rangle$. So the inequality $\varphi_F(x,x^*)<\langle x^*,x\rangle$ is not possible, thus $\langle x^*,x\rangle\le\varphi_F(x,x^*)$ for all $(x,x^*)\in X\times X^*$.

In order to show the second assertion, let $\langle x^*,x\rangle=\varphi_F(x,x^*)$. We already showed that this implies $\langle x^*,y-x\rangle\le F\left(x,y\right)$ for all $y\in X$ which means that $x^*\in A^F\left(x\right)$.

Conversely, assume that $x^*\in A^F\left(x\right)$; then $\langle x^*,y-x\rangle\le F\left(x,y\right)$ for all $y\in X$. By monotonicity of F we obtain $\langle x^*,y-x\rangle\le -F\left(y,x\right)$ for all $y\in X$. This implies that $\langle x^*,y\rangle+F(y,x)\le\langle x^*,x\rangle$ for all $y\in X$. From here we conclude that $\varphi_F(x,x^*)\le\langle x^*,x\rangle$. By the first part of the proof, $\varphi_F(x,x^*)=\langle x^*,x\rangle$. The proof is complete. □

The Fitzpatrick transform of a monotone bifunction and the Fitzpatrick function of an operator are related via the following proposition.

Proposition 3.3: *Let T be a monotone operator. Then $\varphi_{G_T}=\mathcal{F}_T$, where $\mathcal{F}_T$ is the Fitzpatrick function of T.*

Proof: For each $(x,x^*)\in X\times X^*$,

$$\begin{aligned}\varphi_{G_T}\left(x,x^*\right)&=\sup_{y\in X}\left(\langle x^*,y\rangle+G_T(y,x)\right)=\sup_{y\in X}\left(\langle x^*,y\rangle+\sup_{y^*\in T(y)}\langle y^*,x-y\rangle\right)\\&=\sup_{(y,y^*)\in\operatorname{gr}T}\left(\langle x^*,y\rangle+\langle y^*,x\rangle-\langle y^*,y\rangle\right)=\mathcal{F}_T\left(x,x^*\right).\end{aligned}$$

□

In a similar way as in [8], given a monotone bifunction F we define on $X\times X^*$ the upper Fitzpatrick transform φ^F of F by

$$\varphi^F(x,x^*)=\sup_{y\in X}(\langle x^*,y\rangle-F(x,y)),\ \forall(x,x^*)\in X\times X^*.$$

Remark 1: It is easy to show that F is BO-maximal monotone if and only if for all $(x,x^*)\in X\times X^*$, the following equivalence holds:

$$\langle x^*,x\rangle\ge\varphi_F(x,x^*)\Longleftrightarrow\langle x^*,x\rangle\ge\varphi^F(x,x^*). \tag{9}$$

In fact, given that $\varphi_F\le\varphi^F$, (9) is equivalent to $\langle x^*,x\rangle\ge\varphi_F(x,x^*)\Longrightarrow\langle x^*,x\rangle\ge\varphi^F(x,x^*)$ or, successively,

$$\langle x^*,x\rangle\ge\sup_{y\in X}\left(\langle x^*,y\rangle+F(y,x)\right)\Rightarrow\langle x^*,x\rangle\ge\sup_{y\in X}(\langle x^*,y\rangle-F(x,y))$$

$$\langle x^*,y-x\rangle+F(y,x)\le 0,\ \forall y\in X\Rightarrow\langle x^*,y-x\rangle\le F(x,y),\ \forall y\in X.$$

The last line means that F is BO-maximal monotone. Note that whenever F is BO-maximal monotone Theorem 3.2 implies $\langle x^*,x\rangle\le\varphi_F(x,x^*)\le\varphi^F(x,x^*)$, so (9) can be rewritten as

$$\langle x^*,x\rangle=\varphi^F(x,x^*)\Longleftrightarrow\langle x^*,x\rangle=\varphi_F(x,x^*). \tag{10}$$

Note also that

$$\varphi^F(x,x^*) = (F(x,\cdot))^*(x^*)$$
$$\varphi_F(x,x^*) = (-F(\cdot,x))^*(x^*)$$

These equalities hold for each $(x,x^*) \in X \times X^*$. In case $x \notin D(F)$, both sides of the first equality are equal to $+\infty$. Also,

$$\begin{aligned}\left(\varphi^F\right)^*(x^*,x) &= \sup_{(y,y^*)\in X\times X^*}\{\langle y^*,x\rangle + \langle x^*,y\rangle - \varphi^F(y,y^*)\}\\ &= \sup_{(y,y^*)\in X\times X^*}\{\langle y^*,x\rangle + \langle x^*,y\rangle - (F(y,\cdot))^*(y^*)\}\\ &= \sup_{y\in X}\{\langle x^*,y\rangle + (F(y,\cdot))^{**}(x)\}\end{aligned}$$

and

$$\begin{aligned}(\varphi_F)^*(x^*,x) &= \sup_{(y,y^*)\in X\times X^*}\{\langle y^*,x\rangle + \langle x^*,y\rangle - \varphi_F(y,y^*)\}\\ &= \sup_{(y,y^*)\in X\times X^*}\{\langle y^*,x\rangle + \langle x^*,y\rangle - (-F(\cdot,y))^*(y^*)\}\\ &= \sup_{y\in X}\{\langle y^*,x\rangle + (-F(\cdot,y))^{**}(x)\}.\end{aligned}$$

In the special case where $F(x,\cdot)$ is convex and lsc for all $x \in D(F)$, then $(F(y,\cdot))^{**} = F(y,\cdot)$ for every $y \in X$, so $\left(\varphi^F\right)^*(x^*,x) = \varphi_F(x,x^*)$. The following theorem, stated for the reflexive case for simplicity, shows that the arguments of [8] can be used in our framework to obtain the following result. As in [8] we will use the following theorem from [9, 14].

Theorem 3.4: *Let X be reflexive. If $h : X \times X^* \to \overline{\mathbb{R}}$ is a proper, lsc and convex function such that $h(x,x^*) \geq \langle x^*,x\rangle$ and $h^*(x^*,x) \geq \langle x^*,x\rangle$, then the operator with graph $\{(x,x^*) : h(x,x^*) = \langle x^*,x\rangle\}$ is maximal monotone.*

Theorem 3.5: *Let X be reflexive, F be BO-maximal monotone and $F(x,\cdot)$ be convex and lsc for all $x \in D(F)$. Then F is maximal monotone.*

Proof: The assumption that $F(x,\cdot)$ is convex and lsc implies that $\left(\varphi^F\right)^*(x^*,x) = \varphi_F(x,x^*)$. Since $\varphi_F \leq \varphi^F$ and φ_F is convex and lsc, we deduce that

$$\varphi^F(x,x^*) \geq \overline{\text{co}}\varphi^F(x,x^*) \geq \varphi_F(x,x^*) = \left(\varphi^F\right)^*(x^*,x) = \left(\overline{\text{co}}\varphi^F\right)^*(x^*,x). \quad (11)$$

By Theorem 3.2 we know that $\varphi_F(x,x^*) \geq \langle x^*,x\rangle$ with equality if and only if $x^* \in A^F(x)$. This shows in particular that all functions appearing in (11) are proper, since $\varphi^F \equiv +\infty$ implies $\left(\varphi^F\right)^* \equiv -\infty$ which is impossible. By Remark **1** in this section, $\varphi^F(x,x^*) = \langle x^*,x\rangle$ if and only if $x^* \in A^F(x)$. Combining with (11) we obtain that $\overline{\text{co}}\varphi^F(x,x^*) \geq \langle x^*,x\rangle$ and $\left(\overline{\text{co}}\varphi^F\right)^*(x^*,x) \geq \langle x^*,x\rangle$, with equality if and only if $x^* \in A^F(x)$. Theorem 3.4 now implies that A^F is maximal monotone. □

Note that it is not necessary to have $F(x,x) = 0$ for all $x \in D(F)$ or to have a closed and convex $D(F)$. Of course, in the case $F(x,y) = +\infty$ when $x \in D(F)$ and $y \notin D(F)$ that was considered in previous papers, the assumption on $F(x,\cdot)$ implies that $D(F)$ is convex.

We will need the following result, which is a simple adaptation of Proposition 4.1 of [12] to our framework. Note that in [12] all bifunctions were supposed to satisfy $F(x,x)=0$, $x\in D(F)$, but this property was actually not needed in Proposition 4.1 that we use.

Proposition 3.6: *Let X be reflexive and F be maximal monotone. Assume that for every $x\in D(F)$ and any converging sequence $\{x_n\}\subseteq D(F)$, the sequence $\{F(x,x_n)\}$ is bounded from below*[1]*. Then $D(F)\subseteq\overline{D(A^F)}$. In particular, $\overline{D(F)}$ is convex.*

Proof: Define the monotone bifunction F_1 by

$$F_1(x,y)=\begin{cases} F(x,y), & x\notin D(F) \text{ or } y\in D(F)\\ +\infty, & x\in D(F) \text{ and } y\notin D(F).\end{cases}$$

Then $A^F(x)\subseteq A^{F_1}(x)$ for all $x\in X$ and by maximal monotonicity of F, $A^F=A^{F_1}$. We apply Proposition 4.1 of [12] and get the result. □

A trivial consequence is the following corollary.

Corollary 3.7: *Assume that X is reflexive, F is maximal monotone and $F(x,\cdot)$ is lsc for every $x\in D(F)$. Then $D(F)\subseteq\overline{D(A^F)}$.*

Using the above, we now show that whenever $F(x,\cdot)$ is convex and lsc, BO-maximal monotonicity is equivalent to a more general statement than (5).

Proposition 3.8: *Let X be reflexive. Assume that F is a monotone bifunction and $F(x,\cdot)$ is convex and lsc for every $x\in D(F)$. Then the following are equivalent:*

(i) F is BO-maximal monotone.

(ii) For each given $\overline{x}\in X$ and for every convex, lsc function ψ with $\psi(\overline{x})=0$ and $\operatorname{int}(\operatorname{dom}\psi)\cap D(F)\neq\emptyset$, the following implication holds:

$$F(y,\overline{x})\le\psi(y),\ \forall y\in D(F)\implies \exists x^*\in\partial\psi(\overline{x}): 0\le F(\overline{x},y)+\langle x^*,y-\overline{x}\rangle,\ \forall y\in X.$$

(iii) For each given $\overline{x}\in X$ and for every convex, lsc function ψ with $\psi(\overline{x})=0$ and $\operatorname{int}(\operatorname{dom}\psi)\cap D(F)\neq\emptyset$, the following implication holds:

$$F(y,\overline{x})\le\psi(y),\ \forall y\in D(F)\implies 0\le F(\overline{x},y)+\psi(y),\ \forall y\in X\ .$$

Proof: Implications (ii)⇒(iii) and (iii)⇒(i) are obvious, so we prove only implication (i)⇒(ii). Let $\overline{x}\in X$ and $\psi(\overline{x})=0$. Suppose that

$$F(y,\overline{x})\le\psi(y)\qquad\forall y\in D(F) \tag{12}$$

Our assumptions on ψ imply that ψ is proper, so $\partial\psi$ is maximal monotone. By Theorem 3.5 the operator A^F is also maximal monotone. By assumption, $\operatorname{int}(\operatorname{dom}\psi)\cap D(F)\neq\emptyset$. Since $\operatorname{int}(\operatorname{dom}\psi)=\operatorname{int}(\operatorname{dom}\partial\psi)$ and $D(F)\subseteq\overline{D(A^F)}$ by Corollary 3.7, we infer that $\operatorname{int}(\operatorname{dom}\partial\psi)\cap D(A^F)\neq\emptyset$. It follows that $A^F+\partial\psi$ is maximal monotone.

For every $y\in D(\partial\psi)\cap D(A^F)$ and every $y_1^*\in A^F(y)$ and $y_2^*\in\partial\psi(y)$, relation (12) implies

$$\langle y_1^*,\overline{x}-y\rangle\le F(y,\overline{x})\le\psi(y)=\psi(y)-\psi(\overline{x})\le-\langle y_2^*,\overline{x}-y\rangle$$

[1]The bound may depend on x.

so

$$\langle y_1^* + y_2^*, y - \overline{x} \rangle \geq 0. \tag{13}$$

Relation (13) can be written as $\langle y^* - 0, y - \overline{x} \rangle \geq 0$ for all $(y, y^*) \in \operatorname{gr}\left(A^F + \partial\psi\right)$. Hence $0 \in (A^F + \partial\psi)(\overline{x})$, i.e., there exists $x^* \in \partial\psi(\overline{x})$ such that $-x^* \in A^F(\overline{x})$. This means that

$$\langle -x^*, y - \overline{x} \rangle \leq F(\overline{x}, y), \ \forall y \in X$$

i.e., (ii) holds. □

The equivalence (i) ⇔ (iii) was proved by other methods in [5], assuming in addition that ψ is continuously Gâteaux differentiable, $D(F)$ is convex and contained in $\operatorname{dom}\psi$, $F(x,x) = 0$ for all $x \in D(F)$, and $F(x,y) = +\infty$ for $x \in D(F)$, $y \notin D(F)$.

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