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Transformation of Quasiconvex Functions to Eliminate Local Minima

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Abstract Quasiconvex functions present some difficulties in global optimization, because their graph contains "flat parts", thus a local minimum is not necessarily the global minimum. In this paper, we show that any lower semicontinuous quasiconvex function may be written as a composition of two functions, one of which is nondecreasing, and the other is quasiconvex with the property that every local minimum is global minimum. Thus, finding the global minimum of any lower semicontinuous quasiconvex function is equivalent to finding the minimum of a quasiconvex function, which has no local minima other than its global minimum.

The construction of the decomposition is based on the notion of "adjusted sublevel set". In particular, we study the structure of the class of sublevel sets, and the continuity properties of the sublevel set operator and its corresponding normal operator.

Keywords Quasiconvex Function · Generalized Convexity · Adjusted Sublevel Sets · Normal Operator

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1 Introduction

Quasiconvexity is one of the most important generalizations of convexity. It was used since the first half of the last century in minimax theorems and in economics, and later on, in optimization [1].

One of the main features of quasiconvex functions is that, in contrast with convex functions, a local minimum might be not global. Because of this, in many cases quasiconvex functions are more difficult to handle, and quite often, additional assumptions are imposed; for example, that the function is semistrictly quasiconvex [2], or pseudoconvex etc.

By definition, a real-valued function is quasiconvex if its lower level sets are convex. It is then natural that several studies of quasiconvex functions are based on properties related to the sublevel sets. For example, Borde and Crouzeix [3] have studied the continuity properties of the normal cone operator to the strict sublevel sets (i.e., the operator whose value at each point is the normal cone to the strict sublevel set defined by the point). Aussel and Daniilidis [4] characterized some classes of quasiconvex functions (quasiconvex, strictly quasiconvex and semistrictly quasiconvex) by the monotonicity properties of the normal cone operator.

It can be shown that the normal cone operator to the sublevel sets of a quasiconvex function is quasimonotone, while the normal cone operator to the strict sublevel sets of a lower semicontinuous function is cone upper semicontinuous [3]. However, neither of the two normal operators has both properties. To remedy this, one may assume for example that the function is semistrictly quasiconvex. A more general approach is to use the so-called adjusted sublevel sets: the normal cone operator to the adjusted sublevel sets is both quasimonotone and cone upper semicontinuous [5], and so it is convenient for studying the optimization of quasiconvex functions which are not necessarily semistrictly quasiconvex. In addition, Aussel and Pistek [6] showed that the sublevel set operator is lower semicontinuous and this is related to the cone upper semicontinuity of the corresponding normal cone operator.

In this paper, our primary aim is to show that every quasiconvex lower semicontinuous function f can be written as a composition $h \circ g$ where h is a nondecreasing function, and g is a neatly quasiconvex function, i.e., a quasiconvex function with the additional property that every point of local minimum is a point of global minimum. The functions f and g have exactly the same points of global minimum, but g has no additional local minima. The construction makes use of the adjusted sublevel sets. Initially we will show that the adjusted sublevel set operator is lower semicontinuous and, as a result, we will recover the cone upper semicontinuity of the corresponding normal cone operator. Then we will study the properties of the class of all adjusted sublevel sets. We will show that this class is totally ordered by inclusion, and we will use it to construct the function g. We expect that this decomposition will permit to study the optimization of general quasiconvex functions starting from the easier case of neatly quasiconvex functions.

2 Preliminaries

For any $x, y \in \mathbb{R}^n$, we set $]x, y[= \{tx + (1-t)y : 0 < t < 1\}$, and $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$. Also, if $\varepsilon > 0$, we denote by $B(x, \varepsilon)$ the open ball $\{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$, and we denote by $\overline{B}(x, \varepsilon)$ the closed ball $\{y \in \mathbb{R}^n : \|y - x\| \leq \varepsilon\}$.

Given a nonempty set $A \subseteq \mathbb{R}^n$, the closure, the boundary, and the complement of A will be denoted by $\bar{A}$, ∂A, and A^c, respectively. The convex hull and the conic hull generated by the set A are denoted by $\mathrm{conv}(A)$ and $\mathrm{cone}(A)$, respectively; $\dim(A)$ denotes the dimension of $\mathrm{conv}(A)$. Also, we set $B(A,\varepsilon) = \{y \in \mathbb{R}^n : \mathrm{dist}(y,A) < \varepsilon\}$, and $\overline{B}(A,\varepsilon) = \{y \in \mathbb{R}^n : \mathrm{dist}(y,A) \leq \varepsilon\}$, where $\mathrm{dist}(x,A) = \inf\{\|x-a\| : a \in A\}$ for any $x \in \mathbb{R}^n$. The relative interior of A will be denoted by $\mathrm{ri}(A)$.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$. The domain of f is the set $\mathrm{dom}(f)= \{x \in \mathbb{R}^n : f(x) < +\infty\}$. We define for any $\lambda \in \mathbb{R} \cup \{+\infty\}$ the sublevel set and the strict sublevel set by, respectively, $S^f_\lambda = \{y \in \mathbb{R}^n : f(y) \leq \lambda\}$ and $S^{f,<}_\lambda = \{y \in \mathbb{R}^n : f(y) < \lambda\}$. Given $x \in \mathbb{R}^n$ we set for simplicity $S_f(x) = S^f_{f(x)}$ and $S^<_f(x) = S^{f,<}_{f(x)}$. Also we set $\rho^f_x = \mathrm{dist}(x, S^<_f(x))$.

The function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is called

– Quasiconvex, if for all $\lambda \in \mathbb{R}$, S^f_λ is convex. Or equivalently, if for all $x, y \in \mathrm{dom} f$,

$$f(z) \leq \max\{f(x), f(y)\} \text{ for all } z \in [x,y]\,.$$

– Semistrictly quasiconvex, if for all $x, y \in \mathrm{dom} f$,

$$f(x) < f(y) \text{ implies } f(z) < f(y) \text{ for all } z \in \,]x,y[\,.$$

Next we will recall the notion of adjusted sublevel set of a function f at a point $x \in \mathbb{R}^n$ [5].

Definition 2.1 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, and $x \in \mathbb{R}^n$. The adjusted sublevel set $S_f^a(x)$ is the set

$$S_f^a(x) = S_f(x) \cap \overline{B}\left(S_f^<(x), \rho_x^f\right)$$

if $x \notin \arg\min(f)$, and $S_f^a(x) = S_f(x)$ otherwise.

Note that, for all $x \in \mathbb{R}^n$ we have

$$S_f^<(x) \subseteq S_f^a(x) \subseteq S_f(x).$$

In [5], the adjusted sublevel sets have proved to be useful to the study of quasiconvex programs. A part of the usefulness comes from their nice continuity properties, as we will see below.

We recall that, a function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is l.s.c. if and only if S_λ is closed for all $\lambda \in \mathbb{R}$. This implies that $S_f(x)$ is closed for all $x \in \mathbb{R}^n$. The converse for this implication is not always true, as we can see in the following example.

Example 2.1 Take f to be the function defined on $\mathbb{R}^n$ by

$$f(x) = \begin{cases} \| x \|, & \| x \| < 1, \\ \\ 5, & \| x \| \geq 1. \end{cases}, \text{ for every } x \in \mathbb{R}^n.$$

Then $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, while f is not l.s.c. at $\partial B(0,1)$.

The assumption that $S_f(x)$ is closed for all $x \in \mathbb{R}^n$ is usually enough for our purpose, so we will use it instead of lower semicontinuity.

The following property of the adjusted sublevel set is an easy consequence of its definition.

Proposition 2.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function such that for each $x \in \mathbb{R}^n$, $S_f(x)$ is closed. Then $S_f^a(x)$ is closed and convex.*

Now we recall the definition of upper semicontinuity and lower semicontinuity for multivalued maps.

Definition 2.2 A multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be upper semicontinuous (u.s.c.) at $x \in \mathbb{R}^n$, if for every open set $V \subseteq \mathbb{R}^n$ such that $T(x) \subseteq V$, there exists an open set $U \subseteq \mathbb{R}^n$, such that $x \in U$ and $T(u) \subseteq V$, for all $u \in U$. We say that T is lower semicontinuous (l.s.c.) at $x \in \mathbb{R}^n$, if for every open set $V \subseteq \mathbb{R}^n$ with $T(x) \cap V \neq \phi$, there exists an open set $U \subseteq \mathbb{R}^n$, such that $x \in U$ and $T(u) \cap V \neq \phi$, for all $u \in U$.

3 Continuity Properties of the Sublevel Set Operators

Our first main result establishes the lower semicontinuity of the operator $S_f^a(\cdot)$.

Theorem 3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be quasiconvex. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then the map $x \rightrightarrows S_f^a(x)$ is l.s.c. on $\mathbb{R}^n$.*

Proof Fix any $x \in \mathbb{R}^n$ and let $V \subseteq \mathbb{R}^n$ be an open set such that $V \cap S_f^a(x) \neq \phi$. We want to find $\varepsilon > 0$, such that for all $y \in B(x, \varepsilon)$, $V \cap S_f^a(y) \neq \phi$.

We consider three cases:

Case (1) $x \in \arg\min f$. Then for every $y \in \mathbb{R}^n$, $S_f^a(x) = \arg\min f \subseteq S_f^a(y)$ so we have trivially $V \cap S_f^a(y) \neq \phi$.

Case (2) $x \notin \arg\min f$ and $\rho_x^f = 0$, i.e., $x \in \overline{S_f^<(x)}$. Suppose for contradiction that for every $k \in \mathbb{N}$, there exists $y_k \in B(x, \frac{1}{k})$, such that $V \cap S_f^a(y_k) = \phi$.

Then $V \cap S^<_f(y_k) = \phi$, so for all $v \in V$, $f(v) \geq f(y_k)$. Obviously $y_k \longrightarrow x$. But $\{y_k\} \subseteq S_f(v)$ which is closed, so $x \in S_f(v)$. Hence $f(v) \geq f(x)$ for all $v \in V$, which implies $V \cap S^<_f(x) = \phi$. This is impossible since the nonempty set $V \cap S^a_f(x)$ is included in $V \cap \overline{B}(S^<_f(x), \rho^f_x) = V \cap \overline{S^<_f(x)}$.

Case (3) $x \notin \arg\min f$ and $\rho^f_x > 0$. Take any $u \in V \cap S^a_f(x)$. Now take any $w \in S^<_f(x)$ and fix a point v in the open segment joining u and w, close enough to u so that $v \in V$. Since the function $\mathrm{dist}(\cdot, S^<_f(x))$ is convex and $\mathrm{dist}(u, S^<_f(x)) \leq \rho^f_x$ while $\mathrm{dist}(w, S^<_f(x)) = 0$, we infer that $\mathrm{dist}(v, S^<_f(x)) < \rho^f_x = \mathrm{dist}(x, S^<_f(x))$. Since $\mathrm{dist}(\cdot, S^<_f(x))$ is also continuous, there exists $\varepsilon > 0$ such that for all $y \in B(x, \varepsilon)$, $\mathrm{dist}(v, S^<_f(x)) < \mathrm{dist}(y, S^<_f(x))$. We want to show that $v \in S^a_f(y)$, for all $y \in B(x, \varepsilon)$. Since $v \in V$, this will show that $V \cap S^a_f(y) \neq \emptyset$ for all $y \in B(x, \varepsilon)$ and will prove the theorem.

By quasiconvexity of f we have that $f(v) \leq \max\{f(w), f(u)\} \leq f(x)$.

Now take any $y \in B(x, \varepsilon)$. Then $\mathrm{dist}(y, S^<_f(x)) > \mathrm{dist}(v, S^<_f(x)) \geq 0$ so $y \notin S^<_f(x)$. This means that $f(x) \leq f(y)$, so $v \in S_f(y)$. If $f(x) < f(y)$, then $v \in S^<_f(y)$ so $v \in S^a_f(y)$. If $f(x) = f(y)$, then $S^<_f(y) = S^<_f(x)$. Since $\mathrm{dist}(v, S^<_f(x)) < \mathrm{dist}(y, S^<_f(x))$ we get $\mathrm{dist}(v, S^<_f(y)) < \rho^f_y$ so $v \in \overline{B}(S^<_f(y), \rho^f_y)$. So $v \in S^a_f(y)$ in all cases. □

A similar result holds for the operator $x \rightrightarrows S^<_f(x)$.

Proposition 3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then the map $x \rightrightarrows S^<_f(x)$ is l.s.c. on $\mathbb{R}^n$.*

The proof follows the same steps as case (2) above, so we omit it. See also [6, Lemma 1] for a result very close to Proposition 3.1.

The map $x \rightrightarrows S_f^a(x)$ is not u.s.c. in general, as seen in the following example.

Example 3.1 Take the function f to be the same as ([3], Example 2.2); $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \max\{x,y\}, & x < 0 \text{ and } y < 0, \\ 0, & x \geq 0 \text{ and } y < 0, \\ y, & \text{elsewhere.} \end{cases}$$

So $S_f^a(0,0) = \{(p,q),\ p,q \leq 0\}$, and $S_f^a(x,y) = \{(t,r) : r \leq y, t \in \mathbb{R}\}$ for every $y > 0$. Now take $V = \{(x,y) : x, y < 1\}$ which is an open neighborhood of $S_f^a(0,0)$. For all U open with $(0,0) \in U$ there exist $(x,y) \in U$ such that $y > 0$ and so $S_f^a(x,y) \nsubseteq V$. This implies that S_f^a is not u.s.c..

Upper semicontinuity does not fit well with cone-valued maps [3,7]. Before we state the definition of cone upper semicontinuity, we recall the definition of conic neighborhood of a cone $L \subseteq \mathbb{R}^n$.

Definition 3.1 A conic neighborhood of a cone $L \subseteq \mathbb{R}^n$ is an open cone $M \subseteq \mathbb{R}^n$, such that $L \subseteq M \cup \{0\}$.

Definition 3.2 A cone-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be cone upper semicontinuous at $x \in \mathbb{R}^n$, if for every conic neighborhood M of $T(x)$ there exists an open neighborhood $U \subseteq \mathbb{R}^n$, such that $x \in U$ and M is a conic neighborhood of $T(u)$ for any $u \in U$.

It is not always easy to prove the cone upper semicontinuity of a map by definition. In the next theorem we will show an equivalent continuity condition on the map that is produced from the intersection of the cone-valued map and the unit sphere.

Given a multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we recall that the domain of T is the set D(T)= $\{x \in \mathbb{R}^n : T(x) \neq \phi\}$. We set

$$S^*(0,1) = \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

Theorem 3.2 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a cone-valued map with closed values. Then T is cone u.s.c. at $x \in \mathrm{D}(T)$ if and only if the map $F(\cdot) = T(\cdot) \cap S^*(0,1)$ is u.s.c. at x.*

Proof Assume that T is cone u.s.c.. Since $S^*(0,1)$ is compact, in order to show that F is u.s.c., it is enough to show that it has closed graph [8, Proposition 2.23]. So let $(x_k, y_k) \in \operatorname{Graph} F$, $k \in \mathbb{N}$, be such that $(x_k, y_k) \to (x, y)$. Assume that $y \notin F(x)$; then $y \notin T(x)$. Since $T(x)$ is closed, there exists $\delta \in]0,1[$ such that $\overline{B}(y,\delta) \cap T(x) = \emptyset$. Let $K = \operatorname{cone}(\overline{B}(y,\delta))$. Then $K \cap T(x) = \{0\}$. The set K^c is an open cone, and $T(x) \subseteq K^c \cup \{0\}$. Consequently, there exists $\varepsilon > 0$ such that for all $x' \in B(x,\varepsilon)$, $T(x') \subseteq K^c \cup \{0\}$.

For k sufficiently large, one has $x_k \in B(x,\varepsilon)$. Thus, $y_k \in T(x_k) \subseteq K^c \cup \{0\}$. It follows that $y_k \notin \overline{B}(y,\delta)$. This contradicts $y_k \to y$, and proves that $\operatorname{Graph} F$ is closed.

Conversely, let F be u.s.c. at x, and M be a conic neighborhood of $T(x)$ (i.e., $T(x) \subseteq M \cup \{0\}$). Then M is a neighborhood of $F(x)$, so there exists a neighborhood of x (say, U) such that

$$F(u) \subseteq M, \text{ for all } u \in U.$$

Hence

$$T(u) \cap S^*(0,1) \subseteq M.$$

But M and $T(u)$ are cones, so

$$T(u) \subseteq M \cup \{0\}, \text{ for all } u \in U.$$

Thus, T is cone u.s.c. at x. □

The equivalence of the cone upper semicontinuity of T with the upper semicontinuity of F, established in Theorem 3.2, was also stated without proof in [3].

The lower semicontinuity of a set-valued map implies the cone upper semicontinuity of its "normal cone operator", as we now show.

Corollary 3.1 *Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a l.s.c. map. Let further $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a cone-valued map defined by*

$$M(x) = \{y \in \mathbb{R}^n : \langle y, z - x\rangle \leq 0 \textit{ for all } z \in A(x)\}, \quad x \in \mathbb{R}^n.$$

Then M is cone u.s.c. on $\mathbb{R}^n$.

Proof By Corollary 1 in [6], the map M has a closed graph. It follows easily that the map $F(\cdot) = M(\cdot) \cap S^*(0,1)$ also has a closed graph. Using again [8, Proposition 2.23], we deduce that F is u.s.c.. Thus, M is cone u.s.c., in view of Theorem 3.2. □

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be any function. Then for any $x \in \mathbb{R}^n$, the normal cone to the adjusted sublevel set $S_f^a(x)$ at x, is by definition the set

$$N_f^a(x) = \{y \in \mathbb{R}^n : \langle y, z - x\rangle \leq 0 \text{ for all } z \in S_f^a(x)\}.$$

The normal cone to the strict sublevel set $S_f^<(x)$ at x, is the set

$$N_f^<(x) = \{y \in \mathbb{R}^n : \langle y, z - x\rangle \leq 0 \text{ for all } z \in S_f^<(x)\}.$$

Combining Corollary 3.1 with Theorem 3.1 and Proposition 3.1 we obtain the following immediate corollary.

Corollary 3.2 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \mathbb{R}^n$. Then*

(i) The map $x \rightrightarrows N_f^a(x)$ is cone u.s.c. on $\mathbb{R}^n$.

(ii) The map $x \rightrightarrows N_f^<(x)$ is cone u.s.c. on $\mathbb{R}^n$.

Part (ii) of the corollary is a generalization of Proposition 2.2 of [3]. Part (i) recovers Proposition 3.5 of [5] in the finite dimensional case, without using any assumption on $\mathrm{int} S_f^a(x)$.

Just as with the usual upper semicontinuity, cone upper semicontinuity of a map T with closed values, implies that T is closed.

Proposition 3.2 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a cone-valued map with closed values. If T is cone u.s.c., then T is closed.*

Proof Since T is cone u.s.c., by Theorem 3.2 the map $F(.) = T(.) \cap S^*(0, 1)$ is u.s.c. with closed values. Thus it is closed [8, p. 41].

Let $\{(x_k, y_k)\}$ be a sequence in Graph(T), such that $(x_k, y_k) \longrightarrow (x, y)$.

If $y = 0$, then trivially $y \in T(x)$ because $T(x)$ is a closed cone, so it contains 0.

If $y \neq 0$, then $\{(x_k, \frac{y_k}{\|y_k\|})\} \subseteq \text{Graph}(F)$ for large k, and $(x_k, \frac{y_k}{\|y_k\|}) \longrightarrow (x, \frac{y}{\|y\|})$. Hence $\frac{y}{\|y\|} \in F(x)$. So $\frac{y}{\|y\|} \in T(x)$, and since $T(x)$ is a cone, $y \in T(x)$.

Thus $y \in T(x)$ in both cases, which implies that T is closed. $\square$

As a consequence of Proposition 3.2 and Corollary 3.2 we obtain the following corollary.

Corollary 3.3 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \mathbb{R}^n$. Then*

(i) The map $x \rightrightarrows N_f^a(x)$ is closed.

(ii) The map $x \rightrightarrows N_f^<(x)$ is closed.

For l.s.c. functions, a similar result to part (ii) of the corollary was proved in ([3], Proposition 2.1). See also [6, Cor. 1].

4 Decomposition of Quasiconvex Functions

Our objective in this section is to write any l.s.c. quasiconvex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ as a composition $f = h \circ g$, where $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a quasiconvex function which does not have any local minimum except the global minimum, and $h : \text{Im}(g) \longrightarrow \mathbb{R}$ is nondecreasing. Semistrictly quasiconvex functions are known to have the property that every local minimum is global. However, the following example shows that, in general g might not be semistrictly quasiconvex.

Example 4.1 Take f to be a function defined on the positive orthant $\mathbb{R}^2_+$ in polar coordinates by $f(r, \theta) = \theta$ for $r > 0$, while $f = 0$ at the origin. f is set to be equal to $+\infty$ outside the positive orthant. This function is quasiconvex and l.s.c.. Assume that $f = h \circ g$ where g is semistrictly quasiconvex and h is nondecreasing. Choose $0 < r_1 < r_2$. For each $\theta \in \left]0, \frac{\pi}{2}\right[$ one has $f(0,\theta) < f(r_2,\theta)$ so $g(0,\theta) < g(r_2,\theta)$. By semistrict quasiconvexity, $g(r_1,\theta) < g(r_2,\theta)$. Note also that for $\theta \neq \theta'$ the intervals $[g(r_1,\theta), g(r_2,\theta)]$ and $[g(r_1,\theta'), g(r_2,\theta')]$ are disjoint since if, say, $\theta < \theta'$ then $f(r_2,\theta) < f(r_1,\theta')$ so $g(r_2,\theta) < g(r_1,\theta')$. Thus we have an uncountable set of disjoint nondegenerate intervals $[g(r_1,\theta), g(r_2,\theta)]$, $\theta \in \left]0, \frac{\pi}{2}\right[$, which is impossible.

We have to replace semistrict quasiconvexity by a weaker notion. As the example shows, we cannot avoid g having on a segment a constant value greater than $\arg\min g$. But we can avoid having an n-dimensional "flat" part on its graph, thus escaping the main inconvenience of quasiconvexity. This will be done through a generalization of the notion of g-pseudoconvexity of Crouzeix et al.

On the other hand, we may replace lower semicontinuity by the weaker assumption that for each $x \in \mathbb{R}^n$, $S_f(x)$ is closed.

According to [9], a function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is called g-pseudoconvex (g stands for geometrically) if it is quasiconvex and for every $x \in \mathbb{R}^n$ with $f(x) > \inf f$, $\operatorname{int} S_f(x) \neq \emptyset$ holds, and the sets $S_f(x)$ and $S_f^<(x)$ have the same interior and the same closure. See also [10].

In search of a more general definition, take f to be quasiconvex. Note the following:

If two convex subsets A and B have the same closure, then they have the same relative interior. Indeed, since they are convex, $\mathrm{ri}A = \mathrm{ri}\overline{A} = \mathrm{ri}\overline{B} = \mathrm{ri}B$. Likewise, if the sets have the same relative interior, then they have the same closure since $\overline{A} = \overline{\mathrm{ri}A} = \overline{\mathrm{ri}B} = \overline{B}$. Thus we adopt the definition:

Definition 4.1 A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called neatly quasiconvex if it is quasiconvex and for every x with $f(x) > \inf f$, the sets $S_f(x)$ and $S_f^<(x)$ have the same closure (or equivalently, the same relative interior).

Proposition 4.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quasiconvex function. Then the following are equivalent:*

(a) f is neatly quasiconvex

(b) For each $x \notin \arg\min f$, $\rho_x^f = 0$

(c) Each local minimum of f is a global minimum.

Proof (a)⇒(b): If f is neatly quasiconvex, then for each $x \notin \arg\min f$, $x \in \overline{S_f(x)} = \overline{S_f^<(x)}$ so $\rho_x^f = 0$.

(b)⇒(c): Assume that $\bar{x} \notin \arg\min(f)$. By assumption $\rho_x^f = 0$, so $\bar{x} \in \overline{S_f^<(\bar{x})}$. Hence, every ball $B(\bar{x}, \varepsilon)$ intersects $S_f^<(\bar{x})$, which implies that $\bar{x}$ is not a local minimum.

(c)⇒(a): If f is quasiconvex, but not neatly quasiconvex, then there exists x with $f(x) > \inf f$ such that $\overline{S_f(x)} \setminus \overline{S_f^<(x)} \neq \phi$. Then there exists $\bar{x} \in S_f(x) \setminus \overline{S_f^<(x)}$ (otherwise, $S_f(x) \subseteq \overline{S_f^<(x)}$, this implies $\overline{S_f(x)} \subseteq \overline{S_f^<(x)}$). Then $\bar{x}$ is a local minimum, not global, which is a contradiction to the assumption (c). □

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function such that $S_f(x)$ is closed $\forall x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and let $\mathcal{C}$ be the class of all adjusted level sets of f. Then $\mathcal{C}$ has the following properties:

1. $\mathcal{C}$ is totally ordered by inclusion. That is, for every $A, B \in \mathcal{C}$, $A \subseteq B$ or $B \subseteq A$ holds. In fact, if $A = S_f^a(x)$ and $B = S_f^a(y)$, then $A \subsetneq B$ iff either $f(x) < f(y)$, or $f(x) = f(y)$ and $\rho_x^f < \rho_y^f$; $A = B$ iff $f(x) = f(y)$ and $\rho_x^f = \rho_y^f$.

   Indeed, assume that $A = S_f^a(x)$ and $B = S_f^a(y)$. If $f(x) < f(y)$, then $S_f^a(x) \subseteq S_f(x) \subseteq S_f^<(y) \subseteq S_f^a(y)$. Since $y \in S_f^a(y)$ but $y \notin S_f(x) \supseteq S_f^a(x)$, we obtain $S_f^a(x) \subsetneq S_f^a(y)$. Similarly, $f(x) < f(y)$ implies $S_f^a(x) \subsetneq S_f^a(y)$. If $f(x) = f(y)$ then $S_f(x) = S_f(y)$ and $S_f^<(x) = S_f^<(y)$. If $\rho_x^f = \rho_y^f$, then obviously $S_f^a(x) = S_f^a(y)$. If, say, $\rho_x^f < \rho_y^f$, then $\overline{B}(S_f^<(x), \rho_x^f) \subseteq \overline{B}(S_f^<(y), \rho_y^f)$ so $S_f^a(x) \subseteq S_f^a(y)$. Since $y \in S_f^a(y)$ but $y \notin \overline{B}(S_f^<(x), \rho_x^f) \supseteq S_f^a(x)$, we get $S_f^a(x) \subsetneq S_f^a(y)$. The assertion is proved.
2. $x \in S_f^a(y)$ iff $S_f^a(x) \subseteq S_f^a(y)$. Indeed, $x \in S_f^a(y)$ implies $x \in S_f(y)$; if $f(x) < f(y)$ then $S_f^a(x) \subsetneq S_f^a(y)$; if $f(x) = f(y)$ then $x \in S_f^a(y) \subseteq \overline{B}(S_f^<(y), \rho_y^f) = \overline{B}(S_f^<(x), \rho_y^f)$ so $\rho_x^f \le \rho_y^f$ and $S_f^a(x) \subseteq S_f^a(y)$. The converse is obvious since $x \in S_f^a(x) \subseteq S_f^a(y)$.
3. For every $x \in \mathbb{R}^n \setminus \arg\min f$, $x \in \partial S_f^a(x)$. Indeed, in this case $\overline{B}(S_f^<(x), \rho_x^f) \neq \emptyset$. Since $x \notin S_f^<(x)$, either $\rho_x^f = 0$ and x belongs to the boundary of $S_f^<(x)$, or $\rho_x^f > 0$ and then again x belongs to the boundary of $\overline{B}(S_f^<(x), \rho_x^f)$. In both cases, the property follows.

4. If $A, B \in \mathcal{C}$, $A \subsetneq B$ and $\mathrm{int}B \neq \emptyset$, then $(\mathrm{int}B)\backslash A \neq \emptyset$. More generally, if $A \subsetneq B$, then also $(\mathrm{ri}B)\backslash A \neq \emptyset$.

   Indeed, let $A = S_f^a(x)$ and $B = S_f^a(y)$. For any $z \in \mathrm{ri}B$, $]y, z[\subseteq \mathrm{ri}B$. Since A is closed and $y \notin A$ (if $y \in S_f^a(x)$ then $S_f^a(y) \subseteq S_f^a(x)$ by Property 2 above), we can take $w \in]y, z[$ sufficiently close to y so that $w \notin A$. This proves that $\mathrm{ri}B\backslash A \neq \emptyset$.

We now use the above-mentioned properties of the class of the adjusted sublevel sets, to construct the decomposition of quasiconvex functions with closed sublevel sets.

Theorem 4.1 *For every quasiconvex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, there exists a neatly quasiconvex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $S_g(x) = S_f^a(x)$ for all $x \in \mathbb{R}^n$, and a nondecreasing function $h : \mathrm{Im}g \to \mathbb{R} \cup \{+\infty\}$ such that $f = h \circ g$.*

Proof The proof is divided into four steps as follows:

– Step(1): We show that there exists an increasing function $k : \mathcal{C} \to \mathbb{R}$, i.e., such that if $A, B \in \mathcal{C}$, $A \subsetneq B$ then $k(A) < k(B)$. Assume first that all elements of $\mathcal{C}$ have nonempty interior. Let α be a continuous, positive function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} \alpha(x) d\mu = 1$ (μ the Lebesgue measure). For each $A \in \mathcal{C}$, set

$$k(A) = \int_A \alpha(x) d\mu + n - 1.$$

  By Property 4 above, k is increasing, i.e., if $A, B \in \mathcal{C}$, $A \subsetneq B$ then $n - 1 < k(A) < k(B) \leq n$.

In the general case, we write $\mathcal{C}_m = \{A \in \mathcal{C} : \dim A = m\}$, $0 \leq m \leq n$. Then by Property 1, all elements of $\mathcal{C}_m$ generate the same affine subspace V_m of dimension m, and have a nonempty interior with respect to V_m. We define an increasing function k_m on $\mathcal{C}_m$, $1 \leq m \leq n$ exactly as before (we only need properties 1 and 4) and set $k_0(A) = 0$ for the unique element of $\mathcal{C}_0$ (if it exists). Finally we define k on $\mathcal{C}$ by $k(A) = k_m(A)$ if $A \in \mathcal{C}_m$. It is clear that k is increasing.

- Step(2): We define a function $g : \mathbb{R}^n \to \mathbb{R}$ such that its lower level sets are the elements of $\mathcal{C}$ as follows. Set for each $x \in \mathbb{R}^n$

$$g(x) = k(S^a_f(x)). \tag{1}$$

  Let us show that $S_g(x) = S^a_f(x)$. Indeed, $y \in S_g(x)$ iff $g(y) \leq g(x)$ or $k(S^a_f(y)) \leq k(S^a_f(x))$. Since $\mathcal{C}$ is totally ordered and k is increasing, we obtain that $S^a_f(y) \subseteq S^a_f(x)$. By Property 2, this happens exactly when $y \in S^a_f(x)$.

  It follows that g is quasiconvex and such that $S_g(x)$ are closed for all $x \in \mathbb{R}^n$.

- Step(3): We show that the function g defined by (1) is neatly quasiconvex. Indeed, according to Proposition 4.1, to check that g is neatly quasiconvex, it is enough to show that for every x, $\rho^g_x = 0$. To see this, assume first that $\rho^f_x > 0$. Let y be the projection of x onto $\overline{S^<_f(x)}$. For each $z \in]x, y[$ we have $z \in S_f(x) \backslash S^<_f(x)$ so $f(z) = f(x)$. Thus $S^<_f(x) = S^<_f(z)$ and we deduce that $\rho^f_x = d(x, \overline{S^<_f(x)}) > d(z, \overline{S^<_f(z)}) = \rho^f_z$. Hence, $x \notin S^a_f(z)$. Thus

$S^a_f(z) \subsetneq S^a_f(x)$ so $g(z) < g(x)$ by (1) since k is strictly increasing. Hence $]x, y[\subseteq S^<_g(x)$ so $x \in \overline{S^<_g(x)}$, i.e., $\rho^g_x = 0$.

Now assume that $\rho^f_x = 0$. Then $x \in \overline{S^<_f(x)}$ so there exists a sequence (x_n) such that $f(x_n) < f(x)$ and $x_n \to x$. Since $x \notin S_f(x_n)$ we also have $x \notin S^a_f(x_n)$. This implies $g(x) > g(x_n)$. Thus, $x_n \in S^<_g(x)$ so $x \in \overline{S^<_g(x)}$ and $\rho^g_x = 0$ also in this case.

– Step(4): We construct the function h. First we note that for each $t \in \operatorname{Im} g$, there exists $x \in \mathbb{R}^n$ such that $g(x) = t$. Define the function $h : \operatorname{Im} g \to \mathbb{R} \cup \{+\infty\}$ by $h(t) = f(x)$. Let us show that the function h is well-defined and nondecreasing; to see this, let x_1, x_2 be such that $g(x_1) \leq g(x_2)$. Then $x_1 \in S_g(x_2)$, so $x_1 \in S^a_f(x_2) \subseteq S_f(x_2)$. Hence, $f(x_1) \leq f(x_2)$. This shows that whenever $g(x_1) = g(x_2) = t$ then $f(x_1) = f(x_2)$ so h is well-defined. It also shows that h is nondecreasing.

  In addition, it is clear that for every x, $h(g(x)) = f(x)$ i.e., $f = h \circ g$.

□

In the previous theorem we assumed that for each $x \in \mathbb{R}^n$, the set $S_f(x)$ is closed. Since $S_g(x) = S^a_f(x)$, it follows from Proposition 2.1 that the function g constructed in the theorem still has this property. The question whether g can be chosen semicontinuous when f is lower semicontinuous, is still open, and possibly requires a different or modified approach.

We give below an example to show the construction of the function g, as in the theorem.

Example 4.2 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} -1 + \sqrt{-x}, & x \leq 0 \\ n, & n < x \leq n+1, \quad n = 0, 1, 2, \dots \end{cases}$$

We choose $\alpha(x) = \frac{1}{\pi(1+x^2)}$. The following table gives $S_f^a(x)$ and $g(x)$ for all possible values of x. Here, $n = 0, 1, 2, \dots$.

x	$S_f^a(x)$	$g(x) = \int_{S_f^a(x)} \alpha(x)dx$
$-(n+1)^2 \leq x < -n^2$	$[x, n]$	$\frac{1}{\pi}(\arctan n - \arctan x)$
$x = 0$	$\{0\}$	0
$n < x \leq n+1$	$[-(n+1)^2, x]$	$\frac{1}{\pi}\left(\arctan x + \arctan (n+1)^2\right)$

One can see that f has discontinuities in the positive real axis, whereas g has discontinuities in the positive and in the negative axis. This is not due to the particular construction used in Theorem 4.1; it would be present in any decomposition $f = h \circ g$ with h nondecreasing and g neatly quasiconvex. To see this, note that $f(0) < f(1) < f(-9)$. Since $h : \operatorname{Im} g \to \mathbb{R} \cup \{+\infty\}$ is nondecreasing, one must have $g(0) < g(1) < g(-9)$. Given that g is neatly quasiconvex, we deduce $g(0) < g(\frac{1}{2}) < g(1) < g(-9)$. Now assume that g is continuous on $]-9, 0]$. Then there exist $a \neq b$ in $[-9, 0]$ such that $g(a) = g(1)$ and $g(b) = g(\frac{1}{2})$. This would mean that $f(a) = h(g(a)) = h(g(1)) = f(1) = 0$ and $f(b) = f(\frac{1}{2}) = 0$ which is not possible, since f is decreasing on $]-\infty, 0]$. Thus, necessarily g is discontinuous on $]-\infty, 0[$. An even more extreme behavior can be seen in Example 4.3 below, where g is necessarily discontinuous, even if f is continuous.

A continuous quasiconvex function g such that every local minimum is global minimum, is necessarily semistrictly quasiconvex. This has been proved in [11, Th. 3.37] based on another result and on the separation theorem. We present here a short, direct proof of this fact:

Proposition 4.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be neatly quasiconvex and continuous. Then f is semistrictly quasiconvex.*

Proof Assume that f is neatly quasiconvex and continuous. Let $x, y \in \operatorname{dom} f$ be such that $f(x) < f(y)$. The set $S_f^<(y)$ is open and convex. Since y cannot be a local minimum, we have $y \in \overline{S_f^<(y)}$. As $x \in S_f^<(y)$, we deduce that $[x, y[\subseteq S_f^<(y)$, i.e., $f(z) < f(y)$ for all $z \in [x, y[$. Thus, f is semistrictly quasiconvex. □

Note that the quasiconvex function f of Example 4.1 that cannot be decomposed as $h \circ g$ with h nondecreasing and g semistrictly quasiconvex, is not continuous. A natural question arises: If f is quasiconvex and continuous, is there a decomposition $f = h \circ g$ such that h is nondecreasing, and g neatly quasiconvex and continuous, thus semistrictly quasiconvex? The answer is no, as shown by the following.

Example 4.3 Consider the function f from Example 3.1. It is quasiconvex and continuous. Assume that we can write $f = h \circ g$ where h is nondecreasing and g is neatly quasiconvex and continuous. Then by Proposition 4.2, g is semistrictly quasiconvex. Let us show that g is constant on the set $A = (\{0\} \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \{0\})$. For any two points $x, y \in A$, set $x_n = x - \frac{1}{n}(1, 1)$, $y_n = y -$

$\frac{1}{n}(1,1)$. Since at least one of the coordinates of the points x, y is 0, we have $f(x_n) = f(y_n) = -\frac{1}{n}$. Then $f(x_n) < f(y_{n+1}) < f(x_{n+2})$ from which we deduce $g(x_n) < g(y_{n+1}) < g(x_{n+2})$. By continuity, $g(x) = g(y)$.

Consider the points $a = (-1,-1)$, $b = (1,0)$ and $c = \left(0, -\frac{1}{2}\right)$. From $f(a) < f(b)$ we obtain $g(a) < g(b)$; by semistrict quasiconvexity, $g(c) < g(b)$. Now for every $t > 0$, $f(0,t) = t > 0 = f(b)$ so $g(0,t) > g(b)$. Taking the limit as $t \to 0$ we find $g(0,0) \geq g(b) > g(c) = g(0,0)$, a contradiction.

5 Conclusions

We have shown that every quasiconvex function with closed sublevel sets, may be written as a composition of an increasing function, and a "neatly quasiconvex" function. Neatly quasiconvex functions are the functions for which every local minimum point is a global minimum point, so they are more convenient in optimization. As a tool to achieve this aim, we studied the structure of the class of all adjusted sublevel sets of quasiconvex functions.

A parallel goal to this one is the study of the continuity properties of the adjusted sublevel set operator, that is, the operator which at each point associates the adjusted sublevel set corresponding to this point. We have shown that this operator is lower semicontinuous. The same conclusion holds for the strict lower level set operator. As a consequence, we have recovered the cone-upper semicontinuity of the normal cone operator that was established previously, without any assumption on the nonemptiness of the interior of the sublevel sets.

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