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On cyclic and n–cyclic monotonicity of bifunctions

M.H. Alizadeh · **M. Bianchi** · **N. Hadjisavvas** ·
R. Pini

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Abstract In the recent literature, the connection between maximal monotone operators and the Fitzpatrick function is investigated. Subsequently, this relation has been extended to maximal monotone bifunctions and their Fitzpatrick transform. In this paper we generalize some of these results to maximal n-cyclically monotone and maximal cyclically monotone bifunctions, by introducing and studying the Fitzpatrick transforms of order n or infinite order for bifunctions.

Keywords Monotone bifunction · maximal monotone operator · cyclically monotone bifunction · Fitzpatrick function · n-cyclic monotonicity

1 Introduction

Given a nonempty subset C of a Banach space X, a monotone bifunction, as defined in [10], is a function $F : C \times C \to \mathbb{R}$ such that

$$F(x,y) + F(y,x) \leq 0, \quad \forall x, y \in C.$$

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M.H. Alizadeh
Islamic Azad University, Noor Branch, Noor, Iran
E-mail: alizadehmohammadh@gmail.com

M. Bianchi
Università Cattolica del Sacro Cuore, Milano, Italy
E-mail: monica.bianchi@unicatt.it

N. Hadjisavvas
University of the Aegean, Greece
E-mail: nhad@aegean.gr

R. Pini
Università degli Studi di Milano–Bicocca, Italy
E-mail: rita.pini@unimib.it

During the last two decades, monotone bifunctions were mainly used in the study of the equilibrium problem, which consists in finding $x_0 \in C$ such that $F(x_0,x) \geq 0$ for all $x \in C$. A large variety of problems such as variational inequalities, fixed point problems, Nash equilibria of cooperative games, saddle point problems, can be seen as particular instances of equilibrium problems, and this explains the great interest which led to several hundreds of papers on the subject. On the other hand, in recent years it became clear that the study of monotone bifunctions is closely linked to the study of monotone operators and may shed new light to their theory [1,4,12,15,18].

A major advance in the theory of (maximal) monotone operators is the introduction and use of the Fitzpatrick function (see [11,19] and the references therein). This approach was also used to the study of n-cyclically monotone operators [5–8,13]. In addition, very recently, Fitzpatrick functions (or "Fitzpatrick transforms") were proved to be a valuable tool in the study of monotone bifunctions [3,12,16,17].

In this paper we will introduce and study the notion of n-cyclic monotonicity and the Fitzpatrick transform of order n for bifunctions, based on some ideas from [2]. We will connect maximal cyclic monotonicity of bifunctions to the maximal cyclic monotonicity of an underlying operator, and generalize some of the results of [5] regarding operators to the more general framework of bifunctions. The plan of the paper is as follows.

The next section contains some notation, and a short review of basic definitions on monotone operators, Fitzpatrick functions and monotone bifunctions. Section 3 introduces n–cyclically monotone bifunctions, the Fitzpatrick transform of order n or infinite order, and BO-maximality, and shows the connections between these notions. Section 4 focuses on the special case of bifunctions of the form G_T which are defined starting from an operator T. We will see that notions of maximal monotonicity for such bifunctions coincide with the corresponding notions of operators. Section 5 contains the main results of the paper. Given a bifunction F, a new bifunction F_∞ can be constructed from it; it is shown that F is cyclically monotone if and only if F_∞ is monotone. Using F_∞, one can show that if F is cyclically monotone, then there exists a function f such that $F(x,y) \leq f(y) - f(x)$; in fact, one has equality $F_\infty(x,y) = f(y) - f(x)$ under mild assumptions. The properties of F_∞ permit to show the equivalence of the BO-maximal cyclic monotonicity of F and the maximal cyclic monotonicity of A^F, an operator defined through F and studied in some recent papers [1,3,12,16,17]. Analogously, for any $n = 2,3,\ldots$, given a bifunction F we define a bifunction F_n, through which results about maximal n-cyclic monotonicity will be obtained.

2 Preliminaries

Let X be a Banach space and X^* its topological dual. In the following we will denote by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty, +\infty\}$, and we will use the convention

$$(+\infty) + (-\infty) = (-\infty) + (+\infty) = -\infty. \tag{1}$$

We will also use the usual zero-sum convention $\sum_{n=1}^{0} = 0$. Moreover, $\mathbb{N} = \{1, 2, \cdots\}$.

Given a multivalued operator $T : X \rightrightarrows X^*$ we recall that its domain and graph are, respectively, the sets $D(T) = \{x \in X : T(x) \neq \emptyset\}$ and $\mathrm{gr}\, T = \{(x,x^*) \in X \times X^* : x^* \in T(x)\}$. In order to give the definition of n–cyclic monotonicity, let us consider a finite sequence of points $x_1, x_2, ..., x_n, x_{n+1} = x_1$ in X that will be called *cycle* of length n.

Definition 1 Let $T : X \rightrightarrows X^*$ be an operator and $n \in \mathbb{N}$. T is called *n–cyclically monotone* if for each cycle $x_1, x_2, ..., x_n, x_{n+1} = x_1$ in X and any $x_i^* \in T(x_i)$, $i = 1, \dots, n$,

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0.$$

The operator is called *cyclically monotone* if it is n–cyclically monotone for all $n \geq 2$.

The notion of 2–cyclically monotone operator coincides with the usual notion of monotone operator [21]. Moreover, if T is n-cyclically monotone, then it is also m-cyclically monotone for each $2 \leq m < n$.

An n–cyclically monotone operator T is called maximal n–cyclically monotone is there does not exist a proper n–cyclically monotone extension of T. A maximal 2–cyclically monotone operator is simply called maximal monotone. Likewise, a cyclically monotone operator is called maximal cyclically monotone if it does not have any cyclically monotone proper extension.

Given an operator $T : X \rightrightarrows X^*$, its *Fitzpatrick function* $\mathscr{F}_T$ is the function $\mathscr{F}_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathscr{F}_T(x,x^*) = \sup_{(y,y^*) \in \mathrm{gr}\, T} \left(\langle x^*, y \rangle + \langle y^*, x \rangle - \langle y^*, y \rangle \right).$$

The Fitzpatrick function is convex and lower semicontinuous (lsc) with respect to the pair of variables (x,x^*). In addition, whenever T is maximal monotone, one has $\mathscr{F}_T(x,x^*) \geq \langle x^*, x \rangle$ with equality if and only is $(x,x^*) \in \mathrm{gr}\, T$. More generally, a convex, lsc function $\mathscr{H}$ on $X \times X^*$ such that $\mathscr{H}(x,x^*) \geq \langle x^*, x \rangle$ on $X \times X^*$, and $(x,x^*) \in \mathrm{gr}\, T$ implies $\mathscr{H}(x,x^*) = \langle x^*, x \rangle$, is called a representative function of T. It is known that whenever T is maximal monotone, $\mathscr{F}_T$ is the smallest representative function of T.

The Fitzpatrick function has been proven to be a valuable tool in the study of maximal monotone operators [11, 19]. Recently, the Fitzpatrick function of order n and the Fitzpatrick function of infinite order were introduced, in connection to the study of maximal n–cyclically monotone and maximal cyclically monotone operators.

Definition 2 [5] Given an operator $T : X \rightrightarrows X^*$ and $n \in \{2, 3, \dots\}$, the Fitzpatrick function of T of order n is defined by

$$\mathscr{F}_{T,n}(x,x^*) = \sup \left(\sum_{i=1}^{n-2} \langle x_i^*, x_{i+1} - x_i \rangle + \left\langle x_{n-1}^*, x - x_{n-1} \right\rangle + \langle x^*, x_1 \rangle \right) \tag{2}$$

where the supremum is taken over all families $(x_1, x_1^*), (x_2, x_2^*), ..., \left(x_{n-1}, x_{n-1}^*\right)$ in $\mathrm{gr}\, T$. The Fitzpatrick function of infinite order $\mathscr{F}_{T,\infty}$ is the supremum of the functions $\mathscr{F}_{T,n}$, $n \in \{2, 3, \dots\}$.

When T is maximal n-cyclically monotone (resp. maximal cyclically monotone), $\mathscr{F}_{T,n}$ (resp., $\mathscr{F}_{T,\infty}$) is a representative function of T. More precisely, the following propositions hold [5].

Proposition 1 *Let $T : X \rightrightarrows X^*$ and $n \in \{2,3,\dots\}$ be given. The following are equivalent:*
(i) T is n–cyclically monotone;
(ii) $\mathscr{F}_{T,n}(x,x^*) \le \langle x^*,x\rangle$ for all $(x,x^*) \in \operatorname{gr} T$;
(iii) $\mathscr{F}_{T,n}(x,x^*) = \langle x^*,x\rangle$ for all $(x,x^*) \in \operatorname{gr} T$.
Likewise, the following are equivalent:
(i') T is cyclically monotone;
(ii') $\mathscr{F}_{T,\infty}(x,x^*) \le \langle x^*,x\rangle$ for all $(x,x^*) \in \operatorname{gr} T$;
(iii') $\mathscr{F}_{T,\infty}(x,x^*) = \langle x^*,x\rangle$ for all $(x,x^*) \in \operatorname{gr} T$.

Proposition 2 *(i) Let $T : X \rightrightarrows X^*$ be maximal n–cyclically monotone and $n \in \{2,3,\dots\}$. Then $\langle x^*,x\rangle \le \mathscr{F}_{T,n}(x,x^*)$, with equality if and only if $(x,x^*) \in \operatorname{gr} T$.*
(ii) Let $T : X \rightrightarrows X^*$ be maximal cyclically monotone. Then $\langle x^*,x\rangle \le \mathscr{F}_{T,\infty}(x,x^*)$, with equality if and only if $(x,x^*) \in \operatorname{gr} T$.

In this paper, we use a slightly generalized notion of bifunction, introduced in [3]. By the term bifunction we understand any function $F : X \times X \to \overline{\mathbb{R}}$. A bifunction F is said to be *normal* if there exists a nonempty set $C \subseteq X$ such that $F(x,y) = -\infty$ if and only if $x \notin C$. The set C will be called the domain of F and denoted by $D(F)$.

Given a normal bifunction F, we define the operator $A^F : X \rightrightarrows X^*$ by

$$A^F(x) = \{x^* \in X^* : F(x,y) \ge \langle x^*, y-x\rangle,\ \forall y \in X\}.$$

Notice that $D(A^F) \subseteq D(F)$, and

$$F(x,x) \ge 0 \quad \forall x \in D(A^F). \tag{3}$$

On the other hand, given an operator T one can define a normal bifunction $G_T : X \times X \to \overline{\mathbb{R}}$ with $D(G_T) = D(T)$ by

$$G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y-x\rangle. \tag{4}$$

Notice that $G_T(x,x) = 0$ for all $x \in D(T)$, and $G_T(x,\cdot)$ is lsc and convex.

Definition 3 Let $F : X \times X \to \overline{\mathbb{R}}$ be a normal bifunction, and $n \in \{2,3,\ldots\}$. F is called *n–cyclically monotone* if for each cycle $x_1, x_2, ..., x_n, x_{n+1} = x_1$ in X,

$$\sum_{i=1}^{n} F(x_i, x_{i+1}) \le 0. \tag{5}$$

F is called *cyclically monotone* if it is n–cyclically monotone for all $n \in \{2,3,\ldots\}$.

Note that in (5) it is enough to consider cycles in the domain of F.

A 2–cyclically monotone bifunction is said to be *monotone*. If F is n–cyclically monotone, then $F(x,x) \le 0$ for every $x \in D(F)$ and, consequently, $F(x,x) = 0$ for every $x \in D(A^F)$.

Remark 1 Unlike the situation for operators, it is not true in general that if F is n–cyclically monotone, then it is m–cyclically monotone for all $2 \leq m < n$; this assertion becomes true if F satisfies $F(x,x) \geq 0$ for all $x \in D(F)$.

It is easy to check that if T is an n–cyclically monotone (resp., cyclically monotone) operator, then G_T is an n-cyclically monotone (resp. cyclically monotone) bifunction. We will also see later (cf. Theorem 2 and Corollary 1) that if F is an n–cyclically monotone (resp., cyclically monotone) bifunction, then A^F is an n-cyclically monotone (resp. cyclically monotone) operator.

A monotone bifunction F is called maximal monotone if A^F is maximal monotone. A broader class of monotone bifunctions is provided via the notion of BO–maximal monotonicity. Here, BO stands for Blum and Oettli [10], who first introduced this notion for real-valued bifunctions defined on a subset C of X.

Definition 4 A monotone bifunction F is said to be BO–maximal monotone if for all $(x,x^*) \in X \times X^*$,

$$F(y,x) + \langle x^*, y-x\rangle \leq 0 \quad \forall y \in D(F) \Longrightarrow \langle x^*, y-x\rangle \leq F(x,y) \quad \forall y \in X,$$

i.e.

$$F(y,x) + \langle x^*, y-x\rangle \leq 0 \quad \forall y \in D(F) \Longrightarrow x^* \in A^F(x).$$

According to Proposition 1 in [4], every maximal monotone bifunction is also BO–maximal monotone. The converse is not true, in general, but it holds under some additional assumptions. To this aim, we recall the following result (see Theorem 3.5 in [3]):

Theorem 1 *Let X be reflexive, F be BO–maximal monotone and $F(x,\cdot)$ be convex and lsc for all $x \in D(F)$. Then F is maximal monotone.*

3 Fitzpatrick transform of order n and n–cyclic monotonicity

The next subsections are devoted to the definition of the Fitzpatrick transform of order n, and to the investigation of its relationship with n–cyclic monotonicity of bifunctions.

3.1 The Fitzpatrick transform of order n

Assume that $F : X \times X \to \overline{\mathbb{R}}$ is a normal bifunction.

Definition 5 The *Fitzpatrick transform of F of order $n \in \{2,3,\dots\}$ at $(x,x^*) \in X \times X^*$* is defined by the following recursion formula

$$\varphi_{F,n}(x,x^*) = \sup_{y \in X} \{\varphi_{F,n-1}(y,x^*) + F(y,x)\}, \tag{6}$$

where $\varphi_{F,1}(x,x^*) = \langle x^*, x\rangle$. The *Fitzpatrick transform of infinite order* is defined by

$$\varphi_{F,\infty} = \sup_{n \geq 2} \varphi_{F,n}. \tag{7}$$

The above definition gives

$$\varphi_{F,2}(x,x^*) = \sup_{y\in X}(\langle x^*,y\rangle + F(y,x)) = \varphi_F(x,x^*),$$

the original definition of Fitzpatrick transform of a normal bifunction given in [3] (see also [12]).

It is clear that in (6) the supremum can be taken over $y \in D(F)$. It is also easy to check by induction that for $n \in \{2,3,\dots\}$, $\varphi_{F,n}$ is given by the following formula:

$$\varphi_{F,n}(x,x^*) = \sup_{x_1,\dots,x_{n-1}\in D(F)} \left[\left(\sum_{i=1}^{n-2} F(x_i,x_{i+1})\right) + F(x_{n-1},x) + \langle x^*,x_1\rangle\right] \tag{8}$$

for all $(x,x^*) \in X \times X^*$. It follows that $\varphi_{F,\infty}$ is given by

$$\varphi_{F,\infty}(x,x^*) = \sup_{\substack{n\geq 2\\ x_1,\dots,x_{n-1}\in D(F)}} \left[\left(\sum_{i=1}^{n-2} F(x_i,x_{i+1})\right) + F(x_{n-1},x) + \langle x^*,x_1\rangle\right].$$

If $F(x,\cdot)$ is lsc and convex, then for each $n \in \{2,3,\dots\}$, $\varphi_{F,n}$ and $\varphi_{F,\infty}$ are also lsc and convex on $X \times X^*$.

Remark 2 The recursive formula (6) together with (3) imply that the sequence $\varphi_{F,n}$ is pointwisely increasing on $D(A^F) \times X^*$, and

$$\varphi_{F,n}(x,x^*) \geq \langle x^*,x\rangle, \quad \forall (x,x^*) \in D(A^F) \times X^*. \tag{9}$$

In addition, if we assume that $F(x,x) \geq 0$ for every $x \in D(F)$, then for every $k \in \mathbb{N}$ and $(x,x^*) \in X \times X^*$,

$$\begin{aligned}
\varphi_{F,k+1}(x,x^*) &= \sup_{x_1,\dots,x_k\in D(F)} \left[\left(\sum_{i=1}^{k-1} F(x_i,x_{i+1})\right) + F(x_k,x) + \langle x^*,x_1\rangle\right]\\
&\geq \sup_{x_1,\dots,x_{k-1}\in D(F)} \left[\left(\sum_{i=1}^{k-2} F(x_i,x_{i+1})\right) + F(x_{k-1},x_{k-1}) + F(x_{k-1},x) + \langle x^*,x_1\rangle\right]\\
&\geq \varphi_{F,k}(x,x^*).
\end{aligned}$$

Therefore,

$$\varphi_{F,n}(x,x^*) \uparrow \varphi_{F,\infty}(x,x^*).$$

The notion of Fitzpatrick transform of order n (respectively, ∞) for normal bifunctions includes, as special case, the Fitzpatrick function of the same order for operators. Indeed, given an operator T, we consider the normal bifunction G_T defined by (4). If at the right-hand side of (2) we take first the supremum with respect to $x_i^* \in T(x_i)$, $i = 1,2,\dots,n-1$, and then with respect to $x_1,x_2,...,x_{n-1}$, we obtain for any $n \in \{2,3,\dots\}$,

$$\begin{aligned}
\mathscr{F}_{T,n}(x,x^*) &= \sup_{x_1,\dots,x_{n-1}\in D(T)} \left[\left(\sum_{i=1}^{n-2} G_T(x_i,x_{i+1})\right) + G_T(x_{n-1},x) + \langle x^*,x_1\rangle\right]\\
&= \varphi_{G_T,n}(x,x^*).
\end{aligned} \tag{10}$$

Likewise, $\mathscr{F}_{T,\infty} = \varphi_{G_T,\infty}$, thereby showing that $T \to \mathscr{F}_T$ and $T \to G_T \to \varphi_{G_T}$ lead to the same bifunction.

On the other hand, by comparing (8) and (2), from the definition of A^F it is easy to see that for any normal bifunction F one has

$$\mathscr{F}_{A^F,n} \le \varphi_{F,n}, \qquad \mathscr{F}_{A^F,\infty} \le \varphi_{F,\infty}. \tag{11}$$

Notice that in (11) the inequality can be strict as the following example shows.

Example 1 Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc and convex function with $\operatorname{dom} f = \{x \in X : f(x) < \infty\}$. Define $F : X \times X \to \overline{\mathbb{R}}$ by

$$F(x,y) = f(y) - f(x).$$

According to our convention (1), F is cyclically monotone and one can easily check that for each $n \in \{2,3,...\}$ and every (x,x^*) in $X \times X^*$,

$$\varphi_{F,n}(x,x^*) = \varphi_{F,\infty}(x,x^*) = f(x) + f^*(x^*)$$

where $f^*(x^*)$ denotes the conjugate function of f.

Choose $f(x) = \frac{\|x\|^2}{2}$ and define $F : X \times X \to \mathbb{R}$ by $F(x,y) = \frac{\|y\|^2 - \|x\|^2}{2}$. In this case we have $A^F = \partial f = J$ where J is the duality mapping of X:

$$J(x) = \{x^* \in X^* : \|x\|^2 = \|x^*\|^2 = \langle x^*, x\rangle\}.$$

Then, using that the convex conjugate of $\frac{\|x\|^2}{2}$ is $\frac{\|x^*\|^2}{2}$, we find

$$\varphi_{F,2}(x,x^*) = \frac{\|x\|^2}{2} + \frac{\|x^*\|^2}{2}.$$

On the other hand it is known ([14] Proposition 4.1) that

$$\mathscr{F}_{J,2}(x,x^*) \le \frac{(\|x\| + \|x^*\|)^2}{4}.$$

Thus $\varphi_{F,2}(x,x^*) - \mathscr{F}_{A^F,2}(x,x^*) \ge \frac{(\|x\| - \|x^*\|)^2}{4} > 0$ if $\|x\| \ne \|x^*\|$.

3.2 n–cyclic monotonicity

In this subsection we will extend some results of the literature concerning n-cyclic monotone operators to n-cyclic monotone bifunctions. The following results and examples show that Proposition 1 can be generalized to the framework of normal bifunctions, but in a weaker form.

Theorem 2 *Let F be a normal bifunction and $n \in \{2,3,\dots\}$ be fixed. Consider the following assertions:*

(i) *F is n-cyclically monotone;*
(ii) $\varphi_{F,n}(x,x^*) \le \langle x^*, x\rangle$ *for all* $(x,x^*) \in \operatorname{gr} A^F$;

(iii) $\varphi_{F,n}(x,x^*) = \langle x^*,x\rangle$ *for all* $(x,x^*) \in \mathrm{gr}A^F$;
(iv) A^F *is n-cyclically monotone.*

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).

Proof To show (i) ⇒ (ii), let $(x,x^*) \in \mathrm{gr}A^F$ and assume that F is n-cyclically monotone. Let $x_1,x_2,\dots,x_{n-1} \in D(F)$ be arbitrary points, and $x \in D(A^F)$. By assumption,

$$\left(\sum_{i=1}^{n-2} F(x_i,x_{i+1})\right) + F(x_{n-1},x) + F(x,x_1) \le 0.$$

On the other hand $x^* \in A^F(x)$ thus $F(x,x_1) \ge \langle x^*, x_1 - x\rangle$. From this and the above inequality we infer that

$$\left(\sum_{i=1}^{n-2} F(x_i,x_{i+1})\right) + F(x_{n-1},x) + \langle x^*,x_1\rangle \le \langle x^*,x\rangle .$$

By taking the supremum over all $x_1,x_2,\dots,x_{n-2} \in D(F)$ we conclude that $\varphi_{F,n}(x,x^*) \le \langle x^*,x\rangle$ on $\mathrm{gr}A^F$.

Implication (ii) ⇒ (iii) is trivial by (9).

To show implication (iii) ⇒ (iv), take $(x_1,x_1^*),(x_2,x_2^*),\dots,(x_n,x_n^*) \in \mathrm{gr}A^F$ and set $x_{n+1} = x_1$. Then

$$\begin{aligned}\sum_{i=1}^{n} \langle x_i^*, x_{i+1}-x_i\rangle &= \sum_{i=1}^{n-2}\langle x_i^*, x_{i+1}-x_i\rangle + \left\langle x_{n-1}^*, x_n - x_{n-1}\right\rangle + \langle x_n^*, x_1 - x_n\rangle \\ &\le \sum_{i=1}^{n-2} F(x_i,x_{i+1}) + F(x_{n-1},x_n) + \langle x_n^*,x_1\rangle - \langle x_n^*,x_n\rangle \\ &\le \varphi_{F,n}(x_n,x_n^*) - \langle x_n^*,x_n\rangle = 0.\end{aligned}$$

So A^F is n-cyclically monotone.

Corollary 1 *Let F be a normal bifunction. Consider the following assertions:*

(i) F *is cyclically monotone;*
(ii) $\varphi_{F,\infty}(x,x^*) \le \langle x^*,x\rangle$ *for all* $(x,x^*) \in \mathrm{gr}A^F$;
(iii) $\varphi_{F,\infty}(x,x^*) = \langle x^*,x\rangle$ *for all* $(x,x^*) \in \mathrm{gr}A^F$;
(iv) A^F *is cyclically monotone.*

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).

Proof The proof follows from Theorem 2, taking into account that $\varphi_{F,\infty} = \sup_n \varphi_{F,n}$, and that an operator is cyclically monotone if and only if it is n-cyclically monotone for all n.

Remark 3 The following examples show that in the case of normal bifunctions, conditions (i)-(iv) are not equivalent.

1. The n-cyclic monotonicity of A^F does not imply the n-cyclic monotonicity of F, i.e. condition (iv) does not imply (i).
   To show this, let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable convex function such that $f(0)=0$ and $f(x)>0$ for all $x\neq 0$. Obviously, $f'(0)=0$. Define $F(x,y)=f(y-x)$. Then $F(x,\cdot)$ is lsc and convex, $F(x,x)=0$ but F is not monotone since $F(x,y)+F(y,x)>0$ unless $x=y$. By definition $A^F(x)=\partial F(x,\cdot)(x)$, hence $A^F(x)$ is the derivative of $f(\cdot-x)$ at the point x, i.e., $A^F(x)=\{0\}$. Thus A^F is n-cyclically monotone, while F is not even monotone.
   Notice that also (iii) does not hold; indeed, every $(x,x^*)\in \mathrm{gr}A^F$ is of the form $(x,0)$. On the other hand, if we take any $y_0\neq x$ then
   $$\varphi_{F,n}(x,0)\geq\varphi_{F,2}(x,0)=\sup_{y\in X}(\langle 0,y\rangle+F(y,x))\geq F(y_0,x)>0=\langle 0,x\rangle\,.$$
2. Condition (iii) does not imply (i).
   Take, for instance, $G(x,y)=xy-x^2y^2$. Then G is not monotone since $G(x,y)+G(y,x)>0$ for small positive x,y. Let us find A^G. Assume that $x^*\in A^G(x)$. For $x\neq 0$ we should have $G(x,y)\geq x^*(y-x)$ hence $xy-x^2y^2\geq x^*(y-x)$. This is clearly impossible for large y, so $A^G(x)=\emptyset$. Now take $x=0$. Then $0=G(0,y)\geq x^*y$ holds for every $y\in\mathbb{R}$ if and only if $x^*=0$. Thus $\mathrm{gr}A^G=\{(0,0)\}$. By the definition of $\varphi_{G,2}$, it is clear that $\varphi_{G,2}(0,0)=0=\langle 0,0\rangle$ so (iii) holds, while (i) does not hold.

3.3 BO-maximal n-cyclically monotone bifunctions

We are now interested in investigating the role played by maximality. We intend to introduce a suitable notion of maximality for bifunctions that is related to the maximality of operators, and implies results similar to Proposition 2.

Let F be n-cyclically monotone for some $n\in\{2,3,\dots\}$. According to Theorem 2, whenever $(x,x^*)\in\mathrm{gr}A^F$, one has $\varphi_{F,n}(x,x^*)\leq\langle x^*,x\rangle$. We rewrite the first of these relations as
$$\langle x^*,y-x\rangle\leq F(x,y)\quad\forall y\in X \tag{12}$$
and the second as
$$\left(\sum_{i=1}^{n-2}F(x_i,x_{i+1})\right)+F(x_{n-1},x)+\langle x^*,x_1-x\rangle\leq 0\quad\forall x_1,x_2,...,x_{n-1}\in D(F). \tag{13}$$

Then Theorem 2 says that for an n-cyclically monotone bifunction F, (12) implies (13); we now use the converse implication to introduce a concept that extends the notion of BO-maximal monotone bifunction.

Definition 6 Let $n\in\{2,3,\dots\}$ be fixed. An n-cyclically monotone bifunction $F: X\times X\to\overline{\mathbb{R}}$ is called *BO-maximal n-cyclically monotone* if for every $(x,x^*)\in X\times X^*$ the following implication holds
$$\left(\sum_{i=1}^{n-2}F(x_i,x_{i+1})\right)+F(x_{n-1},x)+\langle x^*,x_1-x\rangle\leq 0\quad\forall x_1,x_2,...,x_{n-1}\in D(F)$$
$$\Longrightarrow\langle x^*,y-x\rangle\leq F(x,y)\quad\forall y\in X,$$

or, equivalently,

$$\varphi_{F,n}(x,x^*) \leq \langle x^*,x\rangle \Rightarrow (x,x^*) \in \operatorname{gr} A^F. \tag{14}$$

Moreover, a cyclically monotone bifunction is called *BO-maximal cyclically monotone* if for every $(x,x^*) \in X \times X^*$,

$$\varphi_{F,\infty}(x,x^*) \leq \langle x^*,x\rangle \Rightarrow (x,x^*) \in \operatorname{gr} A^F. \tag{15}$$

It is clear that a bifunction F is BO-maximal 2–cyclically monotone if an only if it is BO-maximal monotone.

Remark 4 If F is cyclically monotone and BO-maximal n-cyclically monotone for some n, then it is BO-maximal cyclically monotone. If F is n-cyclically monotone and BO-maximal m-cyclically monotone for some $m < n$, and $F(x,x) \geq 0$ for all $x \in D(F)$, then it is also BO-maximal n-cyclically monotone.

It is worthwhile noticing that, even if $F(x,x) \geq 0$ for $x \in D(F)$, one cannot infer the BO-maximal m-cyclic monotonicity of the bifunction from its BO-maximal n-cyclic monotonicity, if $n > m$, as showed by the following

Example 2 Let T be an operator which is maximal 3-cyclically monotone but not maximal monotone in a reflexive space X (such operators exist by Example 2.16 in [5]). Consider the normal bifunction $F = G_T$. It is not hard to see that F is 3-cyclically monotone, and that $\operatorname{gr} T \subseteq \operatorname{gr} A^F$. Furthermore, by Theorem 2, A^F is 3-cyclically monotone, therefore, by maximality, $T = A^F$. By the subsequent Proposition 3, F is BO-maximal 3-cyclically monotone. It is not maximal monotone since, by construction, $A^F = T$ is not maximal monotone. In addition, $F(x,\cdot)$ is convex and lsc. Hence F is not BO-maximal monotone, since, otherwise, it would be maximal monotone by Theorem 1.

The following proposition generalizes the relationship existing between BO-maximal monotonicity and maximal monotonicity.

Proposition 3 *(i) Let F be an n-cyclically monotone bifunction for some $n \in \{2,3,\dots\}$. If A^F is maximal n-cyclically monotone, then F is BO-maximal n-cyclically monotone.*

(ii) Let F be cyclically monotone. If A^F is maximal cyclically monotone, then F is BO-maximal cyclically monotone.

Proof (i) Suppose that $\varphi_{F,n}(x,x^*) \leq \langle x^*,x\rangle$ for some $(x,x^*) \in X \times X^*$. From (11), $\mathscr{F}_{A^F,n}(x,x^*) \leq \langle x^*,x\rangle$. Proposition 2 now implies that $(x,x^*) \in \operatorname{gr} A^F$, thus F is BO-maximal monotone.

(ii) This is again an immediate consequence of (11) and Proposition 2.

The converse of Proposition 3 is false as the following examples show (see also [4] for the case $n = 2$).

Example 3 Let $f:\mathbb{R}\to\mathbb{R}$ be the function

$$f(x)=\begin{cases}1-x^2, & x\in(-1,1)\\ 0, & x\notin(-1,1)\end{cases}$$

and $F:\mathbb{R}^2\to\mathbb{R}$ be the normal bifunction defined by $F(x,y)=f(y)-f(x)$. Then F is cyclically monotone thus, in particular, n-cyclically monotone. It is trivial to see that it is BO-maximal monotone. Since $F(x,x)=0$, by Remark 4 it is BO-maximal n-cyclically monotone for every $n=2,3,\ldots$. An easy calculation shows that

$$\mathrm{gr}A^F=\{(x,0):x\notin(-1,1)\}.$$

Thus A^F is not maximal n-cyclically monotone for any $n=2,3,\ldots$.

A similar example with $F(x,\cdot)$ convex but not lsc, is to take

$$f(x)=\begin{cases}0, & x\in(-1,1)\\ 1, & x\in\{-1,1\}\\ +\infty, & x\notin[-1,1]\end{cases}$$

and define $F(x,y)=f(y)-f(x)$ (with the convention $+\infty-\infty=-\infty$). Then again F is BO-maximal n-cyclically monotone for every $n=2,3,\ldots$, but $A^F(x)=\partial f(x)$ so $\mathrm{gr}A^F=\{(x,0):x\in(-1,1)\}$ and A^F is not maximal n-cyclically monotone for any $n=2,3,\ldots$.

Notice that, in the examples above, $F(x,\cdot)$ is not, at the same time, both lsc and convex. On the analogy of Theorem 1, one could expect that if F is BO-maximal n-cyclically monotone, $F(x,\cdot)$ is lsc and convex, then A^F is maximal n-cyclically monotone, for any $n\in\{3,\ldots\}$. Indeed this implication is true for the special bifunction G_T, as we will see in Section 4. We will discuss the general case in Section 5.

The next proposition generalizes Proposition 2 to BO-maximal n-cyclically monotone bifunctions, and also extends Theorem 3.2 proved in [3].

Proposition 4 *Let $n\in\{2,3,\ldots\}$ be fixed, and assume that F is BO-maximal n-cyclically monotone. Then*

(i) $\langle x^*,x\rangle\le\varphi_{F,n}(x,x^*)$ *for all* $(x,x^*)\in X\times X^*$*;*
(ii) $\langle x^*,x\rangle=\varphi_{F,n}(x,x^*)$ *if and only if* $x^*\in A^F(x)$.

In addition, if F is BO-maximal cyclically monotone, then

(i') $\langle x^*,x\rangle\le\varphi_{F,\infty}(x,x^*)$ *for all* $(x,x^*)\in X\times X^*$*;*
(ii') $\langle x^*,x\rangle=\varphi_{F,\infty}(x,x^*)$ *if and only if* $x^*\in A^F(x)$.

Proof Assume that (i) does not hold; then $\varphi_{F,n}(x,x^*)<\langle x^*,x\rangle$ for some $(x,x^*)\in X\times X^*$, so by (14) we obtain $(x,x^*)\in\mathrm{gr}A^F$. Then we get a contradiction with Theorem 2.

To show (ii), we remark first that if $\langle x^*,x\rangle=\varphi_{F,n}(x,x^*)$, then (14) implies $x^*\in A^F(x)$. Conversely, if $x^*\in A^F(x)$, then Theorem 2 implies immediately $\langle x^*,x\rangle=\varphi_{F,n}(x,x^*)$.

The proof of (i') and (ii') follows the same line of (i) and (ii), taking into account (15) and Corollary 1.

4 The special case of the bifunction G_T

In the previous section we showed that BO-maximal n-cyclic monotonicity of F does not imply the maximal n-cyclic monotonicity of A^F. Now we prove that these two conditions are equivalent in the special case of the normal bifunction G_T associated to a maximal n-cyclically monotone operator T.

Given an operator $T : X \rightrightarrows X^*$, let us define the operator $\overline{\text{co}}\,T : X \rightrightarrows X^*$ by $\overline{\text{co}}\,T(x) = \overline{\text{co}\,T(x)}$, where the closure of the set is meant with respect to the weak* topology.

The operator $\overline{\text{co}}\,T$ has the following properties:

$$G_{\overline{\text{co}}\,T} = G_T, \qquad \mathscr{F}_{T,n} = \mathscr{F}_{\overline{\text{co}}\,T,n}, \qquad \overline{\text{co}}\,T = A^{G_T}.$$

The first of these properties is obvious; the second is a consequence of the equalities $\mathscr{F}_{T,n} = \varphi_{G_T,n} = \varphi_{G_{\overline{\text{co}}\,T},n} = \mathscr{F}_{\overline{\text{co}}\,T,n}$. As for the third, we note first that for every operator T, $T \subseteq A^{G_T}$ holds, thus in particular $\overline{\text{co}}\,T \subseteq A^{G_{\overline{\text{co}}\,T}} = A^{G_T}$. On the other hand, if $x_0^* \in A^{G_T}(x)$ for some $x \in X$, then for all $y \in X$,

$$G_T(x,y) = G_{\overline{\text{co}}\,T}(x,y) = \sup_{x^* \in \overline{\text{co}}\,T(x)} \langle x^*, y-x\rangle \geq \langle x_0^*, y-x\rangle .$$

It then follows from the separation theorem that $x_0^* \in \overline{\text{co}}\,T(x)$, so $A^{G_T} \subseteq \overline{\text{co}}\,T$.

We are now in the position to prove the following result that extends Proposition 2.4 in [3].

Theorem 3 *Let $T : X \rightrightarrows X^*$ be an operator and $n \in \{2,3,\dots\}$ be fixed. The following are equivalent:*

(i) $\overline{\text{co}}\,T$ is maximal n-cyclically monotone;
(ii) T is n-cyclically monotone and the following implication holds:

$$\mathscr{F}_{T,n}(x,x^*) \leq \langle x^*, x\rangle \Rightarrow (x,x^*) \in \text{gr}\,\overline{\text{co}}\,T; \tag{16}$$

(iii) G_T is BO-maximal n-cyclically monotone.

Proof First, it can be easily proved that T is n-cyclically monotone if and only if $\overline{\text{co}}\,T$ is n-cyclically monotone (see also Proposition 2.5 and 2.8 in [9]).

To show that (i) implies (ii), let $\mathscr{F}_{T,n}(x,x^*) = \mathscr{F}_{\overline{\text{co}}\,T,n}(x,x^*) \leq \langle x^*, x\rangle$. Then, from Proposition 2 we deduce that $(x,x^*) \in \text{gr}\,\overline{\text{co}}\,T$

If (ii) holds, then for every cycle $x_1, x_2, \dots, x_n, x_{n+1} = x_1 \in D(T)$ we have

$$\sum_{i=1}^n G_T(x_i, x_{i+1}) = \sum_{i=1}^n \sup_{x_i^* \in T(x_i)} \langle x_i^*, x_{i+1} - x_i\rangle \leq 0$$

since T is n-cyclically monotone. Hence, G_T is n-cyclically monotone.
Moreover, if $\varphi_{G_T,n}(x,x^*) \leq \langle x^*, x\rangle$, then (10) entails $\mathscr{F}_{T,n}(x,x^*) \leq \langle x, x^*\rangle$ and by assumption $(x,x^*) \in \text{gr}\,\overline{\text{co}}\,T = \text{gr}A^{G_T}$, thereby showing that G_T is BO-maximal n-cyclically monotone.

Finally, let us show that (iii) implies (i). Since G_T is n-cyclically monotone, by Theorem 2 we have that $A^{G_T} = \overline{\text{co}}\,T$ is n-cyclically monotone. If $\overline{\text{co}}\,T$ is not maximal, there exists an n-cyclically monotone operator B such that $\operatorname{gr} B \supseteq \operatorname{gr} \overline{\operatorname{co}} T$. Let $(x,x^*) \in \operatorname{gr} B \setminus \operatorname{gr} \overline{\operatorname{co}} T$. By Proposition 2.7 in [5] we have that $\mathscr{F}_{\overline{\operatorname{co}} T,n}(x,x^*) \leq \langle x^*, x\rangle$. This implies that

$$\mathscr{F}_{\overline{\operatorname{co}} T,n}(x,x^*) = \mathscr{F}_{T,n}(x,x^*) = \varphi_{G_T,n}(x,x^*) \leq \langle x^*, x\rangle$$

and from the assumption of BO-maximality of G_T , we deduce that $(x,x^*) \in \operatorname{gr} A^{G_T} = \overline{\operatorname{co}} T$, a contradiction. Therefore $\overline{\operatorname{co}} T$ is maximal n-cyclically monotone.

We note that whenever T is maximal n-cyclically monotone, then $\operatorname{gr} T \subseteq \operatorname{gr} \overline{\operatorname{co}} T$ implies $\overline{\operatorname{co}} T = T$, so (i), (ii) and (iii) in the above proposition hold.

As a particular case of the same proposition, we see that an operator T with weak*-closed convex values is maximal n-cyclically monotone if and only if G_T is BO-maximal n-cyclically monotone.

5 BO-maximality vs maximality

One purpose of this section is to extend Theorem 1 to the case of BO-maximal cyclically monotone bifunctions. As we will see, the reflexivity of the space X is no longer necessary. Furthermore, we also prove a result concerning BO-maximal n-cyclically monotone bifunctions. To do so, for every normal bifunction F we will define the normal bifunctions F_n, $n = 2,3,\ldots$ and F_∞ and obtain some of their properties.

Given a normal bifunction F, let us define the normal bifunctions

$$F_n(x,y) = \sup \sum_{i=1}^{n-1} F(x_i,x_{i+1}), \quad n = 2,3,\ldots$$
$$F_\infty(x,y) = \sup_{n=2,3,\ldots} F_n(x,y),$$

where the first supremum is taken over all families $x_1,\ldots,x_n \in X$ such that $x_1 = x$ and $x_n = y$. Note that $F_2 = F$. Obviously, the families can be taken with $x_2,\ldots,x_{n-1} \in D(F)$. Finally, it is clear that $D(F_n) = D(F_\infty) = D(F)$. It is worth noticing that for every $(x,y) \in \big(D(A^F) \times X\big) \cup \big(X \times D(A^F)\big)$ the sequence $\{F_n(x,y)\}_n$ is pointwisely increasing in view of relation (3); the same is true for every $(x,y) \in X \times X$ if we assume that $F(x,x) \geq 0$ whenever $x \in D(F)$.
If $F(x,\cdot)$ is convex and lsc for $x \in D(F)$, then the functions $F_n(x,\cdot)$, $n = 2,3,\ldots$, and $F_\infty(x,\cdot)$ inherit the same properties.

Concerning the Fitzpatrick transform, it is clear from the definitions that

$$\varphi_{F,n} = \varphi_{F_n,2} = \varphi_{F_n} \qquad \text{for all } n = 2,3,\cdots \tag{17}$$

and also

$$\varphi_{F,\infty} = \varphi_{F_\infty,2} = \varphi_{F_\infty}. \tag{18}$$

In the following propositions we shed some light on the relationships between the monotonicity of F, F_n and F_∞. We first establish a simple property of F_∞.

Lemma 1 *For every $x,y,z \in X$,*

$$F_\infty(x,y) + F_\infty(y,z) \le F_\infty(x,z).$$

More generally, for every $x_1,\dots,x_n \in X$,

$$\sum_{i=1}^{n-1} F_\infty(x_i,x_{i+1}) \le F_\infty(x_1,x_n).$$

Proof For every $n,m \in \mathbb{N}$, $x_2,\dots,x_{n-1} \in X$ and $x_{n+1},\dots,x_{n+m-1} \in X$ we set $x_1 = x$, $x_{n+m} = z$ and $x_n = y$; then

$$\sum_{i=1}^{n-1} F(x_i,x_{i+1}) + \sum_{i=n}^{n+m-1} F(x_i,x_{i+1}) \le F_\infty(x,z).$$

By taking suprema over $x_2,\dots,x_{n-1}$ and $x_{n+1},\dots,x_{n+m-1}$ we get the the first part of the lemma. The second part follows by induction.

Proposition 5 *Given a normal bifunction F, the following are equivalent:*
(i) F is cyclically monotone;
(ii) F_∞ is monotone;
(iii) F_∞ is cyclically monotone.

Proof (i)⇒(ii) For every $x,y \in D(F)$ one has $F_\infty(x,y) + F_\infty(y,x) \le F_\infty(x,x)$ by Lemma 1. Thus, it is enough to show that $F_\infty(x,x) \le 0$. For every cycle $x = x_1, x_2, \dots, x_{n-1}, x_n = x$ in $D(F)$ we get by cyclic monotonicity

$$\sum_{i=1}^{n-1} F(x_i,x_{i+1}) \le 0.$$

By taking the supremum over $x_2,\dots,x_{n-1}$ we obtain $F_\infty(x,x) \le 0$.

(ii)⇒(iii) If F_∞ is monotone, then for every cycle of length n, $x_1,\dots,x_{n+1} = x_1 \in D(F)$, we get in virtue of Lemma 1:

$$\sum_{i=1}^{n} F_\infty(x_i,x_{i+1}) = \sum_{i=1}^{n-1} F_\infty(x_i,x_{i+1}) + F_\infty(x_n,x_1) \le F_\infty(x_1,x_n) + F_\infty(x_n,x_1) \le 0,$$

that is, F is cyclically monotone.

(iii)⇒(i) If F_∞ is cyclically monotone, then clearly F is cyclically monotone since $F \le F_\infty$.

Remark 5 The proposition above shows that whenever F is cyclically monotone, then $F_\infty(x,y) \in \mathbb{R}$ for $x,y \in D(F)$; this is interesting because it is not a priori evident that $F_\infty(x,y) \ne +\infty$ for $x,y \in D(F)$.

Proposition 6 *Let $n \ge 2$. Then F is $(2n-2)$-cyclically monotone if and only if F_n is monotone.*

Proof If F is $(2n-2)$-cyclically monotone, then for every $x,y \in D(F)$, every $x_2,\ldots,x_{n-1} \in D(F)$ and $x_{n+1},\ldots,x_{2n-2} \in D(F)$, if we set $x_1 = x$, $x_n = y$ and $x_{2n-1} = x$ we get

$$\sum_{i=1}^{n-1} F(x_i,x_{i+1}) + \sum_{i=n}^{2n-2} F(x_i,x_{i+1}) = \sum_{i=1}^{2n-2} F(x_i,x_{i+1}) \leq 0.$$

Thus,

$$\sup_{x_2,\ldots,x_{n-1}\in D(F)} \sum_{i=1}^{n-1} F(x_i,x_{i+1}) + \sup_{x_{n+1},\ldots,x_{2n-2}\in D(F)} \sum_{i=n}^{2n-2} F(x_i,x_{i+1}) \leq 0$$

which implies $F_n(x,y)+F_n(y,x) \leq 0$, thereby showing that F_n is monotone.

Conversely, if F_n is monotone, then for every cycle of length $(2n-2)$, $x_1,\ldots,x_{2n-1} = x_1 \in D(F)$, we get

$$\sum_{i=1}^{2n-2} F(x_i,x_{i+1}) = \sum_{i=1}^{n-1} F(x_i,x_{i+1}) + \sum_{i=n}^{2n-2} F(x_i,x_{i+1}) \leq F_n(x_1,x_n) + F_n(x_n,x_1) \leq 0$$

that is, F is $(2n-2)$-cyclically monotone.

Let us now investigate the relationship between BO-maximal monotonicity of F, F_n and F_∞.

Proposition 7 *If F be a BO-maximal cyclically monotone bifunction, then F_∞ is BO-maximal monotone. Moreover, let $n \geq 2$. If F is a $(2n-2)$-cyclically monotone and BO-maximal n-cyclically monotone bifunction, then F_n is BO-maximal monotone.*

Proof By Proposition 5 we know that the cyclic monotonicity of F implies the monotonicity of F_∞. Suppose now that $\varphi_{F_\infty}(x,x^*) \leq \langle x,x^*\rangle$. Since $\varphi_{F,\infty}(x,x^*) = \varphi_{F_\infty}(x,x^*)$, by the assumption on F we get $x^* \in A^F(x)$. But $F \leq F_\infty$ implies $\mathrm{gr} A^F \subseteq \mathrm{gr} A^{F_\infty}$, and thus $x^* \in A^{F_\infty}(x)$, i.e. F_∞ is BO-maximal monotone.

In order to prove the second part, notice that the bifunction F_n is monotone by Proposition 6. Suppose that $\varphi_{F_n}(x,x^*) \leq \langle x,x^*\rangle$. Since $\varphi_{F_n}(x,x^*) = \varphi_{F,n}(x,x^*)$, by the assumption of BO-maximal n-cyclic monotonicity we get $x^* \in A^F(x)$ and in particular $x \in D(A^F)$. Moreover, $F(x,y) \leq F_n(x,y)$ for every $y \in X$, since the sequence $\{F_n(x,y)\}$ is pointwisely increasing on $(x,y) \in D(A^F) \times X$. It follows that $x^* \in A^{F_n}(x)$, hence F_n is BO-maximal monotone.

The following proposition gives a simple necessary and sufficient condition for a normal bifunction to be cyclically monotone.

Proposition 8 *Let $F : X \times X \to \overline{\mathbb{R}}$ be a normal bifunction. The following assertions are equivalent:*

(a) F is cyclically monotone

(b) there exists a proper function $f : X \to \overline{\mathbb{R}}$ with $D(F) \subseteq \mathrm{dom} f$ such that

$$F_\infty(x,y) \leq f(y) - f(x), \quad \forall x,y \in X \tag{19}$$

(c) there exists a proper function $f : X \to \overline{\mathbb{R}}$ with $D(F) \subseteq \operatorname{dom} f$ such that

$$F(x,y) \leq f(y) - f(x), \quad \forall x,y \in X \tag{20}$$

Actually, a proper function f satisfies (19) if and only if it satisfies (20). If in addition $F(x,\cdot)$ is lsc and convex for all $x \in D(F)$ then f in (19) and (20) can be chosen to be lsc and convex.

Proof (a)⇒(b). Choose $x_0 \in D(F)$ and set $f(x) = F_\infty(x_0,x)$. It is clear that f is proper and $f(x)$ is real for all $x \in D(F)$ since F_∞ is monotone, hence $D(F) \subseteq \operatorname{dom} f$. By Lemma 1, for every $x,y \in X$ we have

$$F_\infty(x_0,x) + F_\infty(x,y) \leq F_\infty(x_0,y). \tag{21}$$

If $x \in D(F)$ then $F_\infty(x_0,x) \in \mathbb{R}$; if $x \notin D(F)$ then $F_\infty(x,y) = -\infty$; hence we deduce from (21) that in all cases

$$F_\infty(x,y) \leq F_\infty(x_0,y) - F_\infty(x_0,x) \tag{22}$$

which is relation (19).

(b)⇒(c) is an immediate consequence of the inequality $F(x,y) \leq F_\infty(x,y)$.

(c)⇒(a) is trivial.

Let us show that if (20) holds for some proper function f, then (19) also holds for the same f. For every $x_2,\dots,x_{n-1} \in D(F)$, if we set $x_1 = x$ and $x_n = y$ we obtain $F(x_i,x_{i+1}) \leq f(x_{i+1}) - f(x_i)$, $i = 1,\dots,n-1$. By adding the inequalities we find

$$\sum_{i=1}^{n-1} F(x_i,x_{i+1}) \leq f(y) - f(x).$$

Taking the supremum over all families $x_2,\dots,x_{n-1} \in D(F)$, and $n \in \{2,3,\dots\}$, we deduce (19).

Finally, if $F(x,\cdot)$ is lsc and convex for all $x \in D(F)$ then obviously $f(\cdot) = F_\infty(x_0,\cdot)$ is also lsc and convex.

The next proposition shows that relation (19) becomes an equality under suitable assumptions.

Theorem 4 *Assume that $F : X \times X \to \overline{\mathbb{R}}$ is a cyclically monotone bifunction. Let f be a proper, lsc and convex function such that (19) holds. If F_∞ is BO-maximal monotone (resp., F is BO-maximal cyclically monotone), then $\partial f = A^{F_\infty}$ (resp., $\partial f = A^F = A^{F_\infty}$) and f satisfying (19) is unique up to a constant. If, in addition, $F(x,x) \geq 0$ and $F(x,\cdot)$ is lsc and convex for all $x \in D(F)$, then*

$$F_\infty(x,y) = f(y) - f(x), \quad \forall x \in D(F), y \in X. \tag{23}$$

Proof Let $(x,x^*) \in \operatorname{gr} \partial f$. For every $y \in X$, relation (19) implies

$$F_\infty(y,x) + \langle x^*, y-x \rangle \leq f(x) - f(y) + \langle x^*, y-x \rangle \leq 0.$$

Since F_∞ is BO-maximal monotone, we obtain

$$\langle x^*, y-x \rangle \leq F_\infty(x,y), \quad \forall y \in X.$$

Hence, $(x,x^*) \in \mathrm{gr}A^{F_\infty}$. Since ∂f is maximal monotone and F_∞ is monotone, we deduce that $A^{F_\infty} = \partial f$. Now assume that g is another proper, convex and lsc function such that

$$F_\infty(x,y) \le g(y) - g(x), \quad \forall x,y \in X.$$

By the preceding proof we have $\partial g = A^{F_\infty}$, so $\partial f = \partial g$. This implies that g differs from f by a constant [20].

Suppose now that F is BO-maximal cyclically monotone. This implies, in particular, that F_∞ is BO-maximal monotone. Moreover, from Proposition 4 applied to both F and F_∞, one gets that $A^F = A^{F_\infty}$.

To show (23), assume that f satisfies (19), $F(x,x) \ge 0$, and $F(x,\cdot)$ is lsc and convex for all $x \in D(F)$. For every $x' \in D(F)$ set $f_{x'}(y) = F_\infty(x',y)$, $y \in X$. Since (22) holds for any choice of x_0, we have

$$F_\infty(x,y) \le f_{x'}(y) - f_{x'}(x).$$

By the preceding part of the proof, $f_{x'}$ differs from f by a (real) constant, depending on x'. Thus there exists a function $g : D(F) \to \mathbb{R}$ such that for all $y \in X$ and $x' \in D(F)$,

$$f(y) = f_{x'}(y) + g(x'). \tag{24}$$

Since F_∞ is monotone, $F_\infty(x',x') \le 0$. Also, $F_\infty(x',x') \ge F(x',x') \ge 0$ so finally $f_{x'}(x') = F_\infty(x',x') = 0$. Putting $y = x'$ in (24) we obtain $g(x') = f(x')$. It follows that for all $x' \in D(F)$ and $y \in X$,

$$F_\infty(x',y) = f_{x'}(y) = f(y) - f(x').$$

According to Proposition 3, if F is cyclically monotone and A^F is maximal cyclically monotone, then F is BO-maximal cyclically monotone. We are now in position to prove that the converse also holds whenever $F(x,\cdot)$ is convex and lsc. In contrast to Theorem 1, this result does not require the space to be reflexive.

Corollary 2 *Assume that $F : X \times X \to \overline{\mathbb{R}}$ is a cyclically monotone bifunction, and $F(x,\cdot)$ is lsc and convex for every $x \in D(F)$. If F is BO-maximal cyclically monotone, then A^F is maximal cyclically monotone. The same conclusion holds if F is BO-maximal n-cyclically monotone for some n, provided $F(x,x) \ge 0$ if $x \in D(F)$.*

Proof If F is BO-maximal cyclically monotone, then by Theorem 4, $A^F = \partial f$ for some proper, lsc and convex function f. Thus A^F is maximal cyclically monotone. If F is BO-maximal n-cyclically monotone for some n then it is also BO-maximal cyclically monotone by Remark 4 and the conclusion follows.

Finally we show the next result regarding the maximal n–cyclic monotonicity of A^F.

Theorem 5 *Let X be reflexive and F be $(2n-2)$-cyclically monotone and BO-maximal n-cyclically monotone ($n \ge 2$). Assume that $F(x,\cdot)$ is convex and lsc for every $x \in D(F)$. Then A^F is maximal n-cyclically monotone.*

Proof By Proposition 7, the function F_n is BO-maximal monotone. This fact has two consequences: first, since $F_n(x,\cdot)$ is lsc and convex, by Theorem 1 it follows that A^{F_n} is maximal monotone. Second, by Proposition 4 applied to F_n, we obtain that $\varphi_{F_n}(x,x^*) = \langle x^*,x\rangle$ iff $(x,x^*) \in \mathrm{gr} A^{F_n}$. However, the same Proposition applied to F gives $\varphi_{F,n}(x,x^*) = \langle x^*,x\rangle$ if and only if $(x,x^*) \in \mathrm{gr} A^F$. Since $\varphi_{F,n}(x,x^*) = \varphi_{F_n}(x,x^*)$, we get that $A^F = A^{F_n}$ is maximal monotone. But A^F is n-cyclically monotone by Theorem 2, hence A^F is maximal n-cyclically monotone.

We conclude this section with a result related to Theorem 3.5 in [5], where the subdifferential of a proper, lsc and convex function is involved.

Proposition 9 *Let $F : X \times X \to \overline{\mathbb{R}}$ be a normal bifunction. If there exists a proper, lsc and convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that*

$$F(x,y) \le f(y) - f(x), \quad \forall (x,y) \in X \times X, \tag{25}$$

then

$$\varphi_{F,n}(x,x^*) \le f(x) + f^*(x^*) \quad \forall (x,x^*) \in X \times X^*, \quad n \in \{2,3,\ldots\},$$

and

$$\varphi_{F,\infty}(x,x^*) \le f(x) + f^*(x^*) \quad \forall (x,x^*) \in X \times X^*. \tag{26}$$

If further F is BO-maximal cyclically monotone, $F(x,x) = 0$ and $F(x,\cdot)$ is lsc and convex for all $x \in D(F)$, and $D(F) = \mathrm{dom}\, f$, then relation (26) is an equality.

Proof For every family $x_1,\ldots,x_{n-1}$ in $D(F)$, $x \in X$, $x^* \in X^*$, $n \in \{2,3,\ldots\}$, we have

$$\sum_{i=1}^{n-2} F(x_i,x_{i+1}) + F(x_{n-1},x) + \langle x^*,x_1\rangle \le f(x) - f(x_1) + \langle x^*,x_1\rangle.$$

By taking the supremum over $x_1,\ldots,x_{n-1}$, we get

$$\begin{aligned}\varphi_{F,n}(x,x^*) &\le \sup_{x_1\in X}\left(f(x) - f(x_1) + \langle x^*,x_1\rangle\right)\\ &= f(x) + \sup_{x_1\in X}\left(\langle x^*,x_1\rangle - f(x_1)\right) = f(x) + f^*(x^*).\end{aligned}$$

By taking the supremum over $n \in \{2,3,\ldots\}$, from (7) we obtain

$$\varphi_{F,\infty}(x,x^*) \le f(x) + f^*(x^*).$$

Now assume that F is BO-maximal cyclically monotone, $F(x,x) = 0$ and $F(x,\cdot)$ is lsc and convex for all $x \in D(F)$, and $D(F) = \mathrm{dom}\, f$. By Theorem 4, (23) holds. It follows that

$$\begin{aligned}\varphi_{F,\infty}(x,x^*) = \varphi_{F_\infty}(x,x^*) &= \sup_{y\in D(F)}\{\langle x^*,y\rangle + F_\infty(y,x)\}\\ &= \sup_{y\in \mathrm{dom}\, f}\{\langle x^*,y\rangle - f(y) + f(x)\} = f(x) + f^*(x^*).\end{aligned}$$

If we set $F = G_{\partial f}$ then all assumptions of Proposition 9 hold, and we recover the first part of Proposition 3.5 in [5].

References

1. Ait Mansour, M., Chbani, Z. Riahi, H.: Recession bifunction and solvability of noncoercive equilibrium problems. Commun. Appl. Anal. **7**, 369–377 (2003)
2. Alizadeh, M. H.: Monotone and Generalized Monotone Bifunctions and their Application to Operator Theory. PhD thesis, Department of Product and Systems Design Engineering, University of the Aegean (2012)
3. Alizadeh, M. H., Hadjisavvas, N.: On the Fitzpatrick transform of a monotone bifunction. Optimization **62**, 693–701 (2013)
4. Alizadeh, M. H., Hadjisavvas, N.: Local boundedness of monotone bifunctions. J. Glob. Optim. **53**, 231–241 (2012)
5. Bartz, S., Bauschke, H. H., Borwein, J., Reich, S., Wang, X.: Fitzpatrick functions, cyclic monotonicity and Rockafellar's antiderivative. Nonlinear Anal. **66**, 1198–1223 (2007)
6. Bartz, S., Reich, S.: Minimal antiderivatives and monotonicity. Nonlinear Anal. **74**, 59–66 (2011)
7. Bartz, S., Reich, S.: Abstract convex optimal antiderivatives. Ann. Inst. H. Poincaré Anal. Non Linéaire **29**, 435–454 (2012)
8. Bauschke, H. H., Wang, X.: A convex-analytical approach to extension results for n-cyclically monotone operators. Set-Valued Anal. **15**, 297–306 (2007)
9. Bauschke, H. H., Wang, X.: An explicit example of a maximal 3-cyclically monotone operator with bizarre properties. Nonlinear Anal. **69**, 2875–2891 (2008)
10. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Student **63**, 123–145 (1994)
11. Borwein, J. M.: Maximal monotonicity via convex analysis. J. Convex Anal. **13**, 561–586 (2006)
12. Boţ, R. I., Grad, S-M.: Approaching the maximal monotonicity of bifunctions via representative functions. J. Convex Anal. **19**, 713–724 (2012)
13. Boţ, R. I., Csetnek, E. R.: On extension results for n-cyclically monotone operators in reflexive Banach spaces. J. Math. Anal. Appl. **367**, 693–698, (2010)
14. Burachik, R. S., Fitzpatrick, S.: On a family of convex functions associated to subdifferentials. J. Nonlinear Convex Anal. **6**, 165–171 (2005)
15. Hadjisavvas, N., Khatibzadeh, H.: Maximal monotonicity of bifunctions. Optimization **59**, 147-160 (2010)
16. Iusem, A. N.: On the maximal monotonicity of diagonal subdifferential operators. J. Convex Anal. **18**, 489–503 (2011)
17. Iusem, A. N., Svaiter, B.F.: On diagonal subdifferential operators in nonreflexive Banach spaces. Set-Valued Anal. **20**, 1–14 (2012)
18. Kassay, G., Reich, S., Sabach, S.: Iterative methods for solving systems of variational inequalities in reflexive Banach spaces. SIAM J. Optim. **21**, 1319–1344 (2011)
19. Penot, J.-P.: The relevance of convex analysis for the study of monotonicity. Nonlinear Anal. **58**, 855–871 (2004)
20. Rockafellar, R. T.: On the maximal monotonicity of subdifferential mappings. Pacific J. Math. **33**, 209–216 (1970)
21. Zeidler, E.: Nonlinear Functional Analysis and its Applications, Vol. II/B, Nonlinear Monotone Operators. Springer, Berlin (1990)