Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

# CARATHÉODORY'S ROYAL ROAD OF THE CALCULUS OF VARIATIONS: MISSED EXITS TO THE MAXIMUM PRINCIPLE OF OPTIMAL CONTROL THEORY

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(Communicated by the associate editor name)

Abstract. The purpose of the present paper is to show that the most prominent results in optimal control theory, the distinction between state and control variables, the maximum principle, and the principle of optimality, resp. Bellman's equation are immediate consequences of Carathéodory's achievements published about two decades before optimal control theory saw the light of day.

1. **Introduction.** The Theory of Optimal Control began with the discovery of the maximum principle.[1] By means of this novel necessary condition, possibilities were disclosed by which, for the first time, optimal solutions for a certain class of control problems could be computed that were of utmost importance for their time. At least, these problems could have been computed if numerical methods for the solution of two-point boundary value problems and adequate computers had been available at that time.

For several reasons, the discovery of the maximum principle has a very interesting history. First, its genesis lies in the commencement of the Cold War. Therefore, it is not astonishing that the analysis of aircombat scenarios was the activator of this research and that the very first publications at RAND Corporation[2] appeared as classified reports. Secondly, two groups of leading mathematicians of the superpowers US and USSR competed against each other, without knowing about each other initially. The protagonists were Magnus R. Hestenes, Rufus P. Isaacs, and Richard

2000 *Mathematics Subject Classification.* Primary: 01A60, 49-03; Secondary: 90-03, 91-03.

*Key words and phrases.* History of Calculus of Variations and Optimal Control, Royal Road of Calculus of Variations, Bellman Equation, Maximum Principle, Carathéodory.

Dedicated to Professor Dr. Dr.h.c.mult. George Leitmann on the occasion of his 85th birthday on May 24, 2010.

[1] Its first formulation by Magnus R. Hestenes [15] in 1950: *Thus, $H$ has a maximum value with repect to $a_h$ along a minimizing curve $C_0$.* Its first general form and proof by Lev S. Pontryagin, Vladimir G. Boltyanski, and Revaz V. Gamkrelidze [4] in 1956: Этот факт является частным случаем следующего общего принципа, который мы называем принципом максимума. (This fact is a special case of the following general principle which we call maximum principle.)

[2] The RAND Corporation (Research ANd Development) — its headquarter is located in Santa Monica, California — is a US non-profit organisation and think tank founded after World War II to give advice to the US Armed Forces.

E. Bellman in the "blue corner" and Lev Semyonovich Pontryagin, Vladimir Grigorevich Boltyanski, and Revaz Valerianovich Gamkrelidze in the "red corner". All members of the blue corner later complained about their missed opportunities.[3, 4, 5] In contrast, the names of all members of the red corner will for ever be connected with the maximum principle. Thirdly and finally, the proof of the maximum principle designated the birth of a new field in applied mathematics which has, and continues to have, a great impact on optimization theory and exciting applications in almost all fields of sciences. Optimal Control Theory is still an extremely active and highly current research area until today.

In short, here is the outcome of the competition; see the comprehensive monograph of Plail [23] and also the extract in Pesch, Plail [22], which has been enriched by some more recent findings: Magnus R. Hestenes was the first who formulated the maximum principle in his famous RAND memorandum RM-100 in 1950. He was the first who also distiguished clearly between state and control variables using different letters for their notation and brought the problem into a formulation convenient for the purpose of control problems later on. However, standing in the tradition of the Chicago School of the Calculus of Variations, he was captured by the close vicinity of the new type of problems to the field of the classical calculus of variations. Therefore he did not see the novelty behind his achievements; his assumptions were too strong.

Also Isaacs (1951), the father of differential game theory [16],[6] and Bellman (1954), the inventor of dynamic programming [1], did not see that the maximum principle was hidden in their new theories, resp. mathematical methods. "Missed opportunities?"[7]

On the other side, a group around the famous topologist Lev S. Pontryagin including Vladimir G. Boltyanski and Revaz V. Gamkrelidze was disappointed that they had lost a race in the field of topology against certain French mathematicians; see Plail [23], p. 174. Therefore, they realigned their research activities radically toward applied mathematics, additionally encouraged by tasks posed to them by the military; see [23], p. 175, and [22]. Moreover, they were not captured by too

[3] Hestenes in a letter to Saunders MacLane: *It turns out that I had formulated what is now known as the general optimal control problem. I wrote it up as a RAND report and it was widely circulated among engineers. I had intended to rewrite the results for publication elsewhere and did so about 15 years later.* See MacLane [20].

[4] Bellman in his autobiography: *I should have seen the application of dynamic programming to control theory several years before. I should have, but I didn't.* See Bellman [2], p. 182.

[5] Isaacs [17], p. 20: *Once I felt that here was the heart of the subject [...] Later I felt that it [...] was a mere truism. Thus in [my book] "Differential Games" it is mentioned only by title. This I regret. I had no idea, that Pontryagin's principle and Bellman's maximal principle (a special case of the tenet [of transition], appearing little later in the RAND seminars) would enjoy such widespread citation.*

[6] For some information on the history of Differential Games (today's preferred name: Dynamic Games), see Breitner [5]. This paper also contains a lot of background information on the early days at RAND Corporation, in particular on the work climate in the RAND seminars.

[7] In the historical session of the Banach Center Conference on 50 Years of Optimal Control in Bedlewo, Poland, on September 15, 2008, much was said in the discussions about "missed opportunities". Revaz Valerianovich Gamkrelidze, for example, said: *My life was a series of missed opportunities, but one opportunity, I have not missed, to have met Pontryagin.* In respect hereof, the term "missed opportunity" has to be understood as reminiscence to that discussion. The author does not intend to use this term as if he would like to criticize that certain things should have be seen earlier. With respect to Carathéodory's Royal Road of the Calculus of Variations we will use the more adapted term "missed exits".

deep a knowledge of the calculus of variations which allowed them to formulate the maximum principle in a very general form. They proved it completely before 1958, see [23], p. 182, and [22], and opened therewith the possibility for solving a whole bunch of new and interesting control problems.

The intention of the present paper is to go further back to the antiquity of the maximum principle, the time before World War II, in particular to Caratheodory's famous royal road of the calculus of variations. This ingenious approach to the calculus of variations gives evidence of an exit, from the royal road that is surprisingly close to the maximum principle. A very early missed opportunity? However, we have to admit that the resulting proof of the maximum principle, obtained directly from Carathéodory's results, is, like Hestenes' proof, also not liberated from the too restrictive assumptions of the calculus of variations and will thus not lead to the general maximum principle of Pontryagin, Boltyanski, and Gamkrelidze. Notwithstanding our approach may be of interest because of historical and educational reasons; compare also Pesch, Bulirsch [21]. We now set to work on ...

2. **Carathéodory's Royal Road of the Calculus of Variations.** and follow, with slight modifications of the notation,[8] Carathéodory's book of 1935 [7], Chapter 12 *"Simple Variational Problems in the Small"* and Chapter 18 *"The Problem of Lagrange"*. The book was later translated into English in two parts [9]. The German edition was last reprinted in 1994; see [10].


FIGURE 1. Constantin Carathéodory — Κωνσταντίνος Καραθεοδωρῆ (1938) (Born: 13 Sept. 1873 in Berlin, Died: 2 Feb. 1950 in Munich, Germany) and Constantin Carathéodory and Thales from Milet on a Greek postage stamp

2.1. **First section of road: heading for a new sufficient condition.** We begin with the description of Carathéodory's Royal Road of the Calculus of Variations[9]

[8]We generally use the same symbols as Carathéodory, but use vector notation instead of his component notation.

[9]Hermann Boerner [3] coined the term "Königsweg der Variationsrechnung" in 1953. He habilitated 1934 under Carathéodory.

directly for Lagrange problems. All of these results had essentially been investigated by Carathéodory [6] already in 1926.

Let us first introduce a $C^1$-curve $x = x(t) = (x_1(t), \ldots, x_n(t))^\top$, $t' \le t \le t''$, in an $(n+1)$-dimensional Euclidian space $\mathscr{R}_{n+1}$. The line elements $(t, x, \dot{x})$ of the curve are regarded as elements of a $(2n+1)$-dimensional Euclidian space, say $\mathscr{S}_{2n+1}$.

Already Carathéodory considered Lagrangian variational problems, that can be regarded as precursors of optimal control problems: Minimize

$$I(x) = \int_{t_1}^{t_2} L(t, x, \dot{x})\,\mathrm{d}t \tag{1}$$

subject to, for the sake of simplicity, fixed terminal conditions $x(t_1) = a$ and $x(t_2) = b$, $t' < t_1 < t_2 < t''$, and subject to the implicit ordinary differential equation

$$G(t, x, \dot{x}) = 0 \tag{2}$$

with a real-valued $C^2$-function $L = L(t, x, \dot{x})$[10] and a $p$-vector-valued $C^2$-function $G = G(t, x, \dot{x})$ with $p < n$, both defined on an open domain $\mathscr{A} \subset \mathscr{S}_{2n+1}$. It is assumed that the Jacobian of $G$ has full rank,

$$\operatorname{rank}\left(\frac{\partial G_k}{\partial \dot{x}_j}\right)_{\substack{k=1,\ldots,p\\ j=1,\ldots,n}} = p\,. \tag{3}$$

Carathéodory's intention was to head first for sufficient conditions, and not till then to derive most of the major necessary conditions ultimately terminating with the Euler-Lagrange equations, almost in a reversed historical succession. Here, we proceed only partly on his road, in particular we are aiming to Carathéodory's form of Weierstrass' necessary condition in terms of the Hamilton function. For the complete road, see Carathéodory's original works already cited. A short compendium can also be found in Pesch, Bulirsch [21].

1st Stage: Definition of extremals. Carathéodory firstly coins the term *extremal* in a different way than today. According to him, an extremal is a weak extremum of the problem (1), (2).[11] Hence, it might be either a so-called *minimal* or *maximal*.

2nd Stage: Legendre-Clebsch condition. Carathéodory then shows the Legendre-Clebsch necessary condition

$$L_{\dot{x}\,\dot{x}}(t, x, \dot{x}) \quad \text{must not be indefinite.}$$

Herewith, positive (negative) regular, resp. singular line elements $(t, x_0, \dot{x}_0) \in \mathscr{A}$ can be characterized by $L_{\dot{x}\,\dot{x}}(t, x_0, \dot{x}_0)$ being positive (negative) definite, resp. positive (negative) semi-definite. Below we assume that all line elements are positive regular. In today's terminology: for fixed $(t, x)$ the map $v \mapsto L(t, x, v)$ has a positive definite Hessian $L_{vv}(t, x, v)$.

[10] The twice continuous differentiability of $L$ w.r.t. all variables will not be necessary right from the start.

[11] In Carathéodory's terminology, any two competing curves $x(t)$ and $\bar{x}(t)$ must lie in a close neighborhood, i.e., $|\bar{x}(t) - x(t)| < \epsilon$ and $|\dot{\bar{x}}(t) - \dot{x}(t)| < \eta$ for positive constants $\epsilon$ and $\eta$. The comparison curve $\bar{x}(t)$ is allowed to be continuous with only a piecewise continuous derivative; in today's terminology $\bar{x} \in PC^1([t_1, t_2], \mathbf{R}^n)$. All results can then be extended to analytical comparison curves, if necessary, by the well-known Lemma of Smoothing Corners.

3rd Stage: Existence of extremals and Carathéodory's sufficient condition. We consider a family of curves which is assumed to cover simply a certain open domain of $\mathscr{R} \subset \mathscr{R}_{n+1}$ and to be defined, because of (3), by the differential equation $\dot{x} = \psi(t, x)$ with a $C^1$-function $\psi$ so that the constraint (2) is satisfied. Carathéodory's sufficient condition then reads as follows.

**Theorem 2.1** (Sufficient condition)**.** *If a $C^1$-function $\psi$ and a $C^2$-function $S(t, x)$ can be determined such that*

$$L(t, x, \psi) - S_x(t, x)\, \psi(t, x) \equiv S_t(t, x)\,, \tag{4}$$

$$L(t, x, x') - S_x(t, x)\, x' > S_t(t, x) \tag{5}$$

*for all $x'$, which satisfy the boundary conditions $x'(t_1) = a$ and $x'(t_2) = b$ and the differential constraint $G(t, x, x') = 0$, where $|x' - \psi(t, x)|$ is sufficiently small with $|x' - \psi(t, x)| \neq 0$ for the associated line elements $(t, x, x')$, $t \in (t_1, t_2)$, then the solutions of the boundary value problem $\dot{x} = \psi(t, x)$, $x(t_1) = a$, $x(t_2) = b$ are minimals of the variational problem (1), (2).*

2.2. **Carathéodory's Exit to Bellman's Equation.** Carathéodory stated verbatim (translated by the author from the German edition of 1935, [7], p. 201) [for the unconstrained variational problem (1)]: *According to this last result, we must, in particular, try to determine the functions $\psi(t, x)$ and $S(t, x)$ so that the expression*

$$L^*(t, x, x') := L(t, x, x') - S_t(t, x) - S_x(t, x)\, x'\,, \tag{6}$$

*considered as a function of $x'$, possesses a minimum for $x' = \psi(t, x)$, which, moreover, has the value zero.* In today's terminology:

$$S_t = \min_{x'} \{L(t, x, x') - S_x\, x'\}\ ; \tag{7}$$

see also the English edition of 1965 ([9], Part 2) or the reprint of 1994 ([10], p. 201). This equation became later known as Bellman's equation and laid the foundation of his Dynamic Programming Principle; see the 1954 paper of Bellman [1].[12]

For the Lagrange problem (1), (2), Eq. (7) reads as

$$S_t = \min_{\substack{x' \text{ such that} \\ G(t,x,x')=0}} \{L(t, x, x') - S_x\, x'\}\ ; \tag{8}$$

compare Carathéodory's book [7] of 1935, p. 349. Carathéodory considered only unprescribed boundary conditions there.

*Proof.* Carathéodory's elegant proof of Theorem 1 is based on so-called equivalent variational problems where, for all $C^2$-functions $S(t.x)$, the variational problems with the integrands $L$, resp. $L^*(t, x, \dot{x}) := L(t, x, \dot{x}) + S_t + S_x\, \dot{x}$ have the same

[12] In Breitner [5], p. 540, there is an interesting comment by W. H. Flemming: *Concerning the matter of priority between Isaacs' tenet of transition and Bellman's principle of optimality, my guess is that these were discovered independently, even though Isaacs and Bellman were both at RAND at the same time … In the context of calculus of variations, both dynamic programming and a principle of optimality are implicit in Carathéodory's earlier work, which Bellman overlooked. There was some level of contention between Isaacs and Bellman, as the following personal remembrance indicates. One day in the early 1950s, Bellman was giving a seminar at RAND in which he solved some optimization problems by dynamic programming. At the end of Bellman's seminar lecture. Isaccs correctly stated that this problem could also be solved by his own methods. Bellman disagreed. After each of the two reiterated his opinion a few times, Isaacs said: "If the Belman says it three times, it must be true." This quote refers to a line from Lewis Carroll's nonsense tail in verse "The Hunting of the Snark". One of the main (and other absurd) characters in this tale is called the Bellman.*

extremals and there also holds $L_{\dot{x}\,\dot{x}} = L^*_{\dot{x}\,\dot{x}}$. In today's terminology, we have added a null Lagrangian, that is the total derivative of a function $S$ that depends on time and space.

Firstly, one immediately sees that

$$\begin{aligned} I(x) - I^*(x) &= \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t))\,\mathrm{d}t - \int_{t_1}^{t_2} L^*(t, x(t), \dot{x}(t))\,\mathrm{d}t \\ &= -\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} S(t, x(t))\,\mathrm{d}t = S(t_1, a) - S(t_2, b) \end{aligned}$$

differs only by a constant depending on $S$ and the end conditions. Hence the minimizers of the original problem (1) coincide with the ones of the equivalent problem associated with $L^*$.

In order to find a method to determine the function $S$, Carathéodory sought for a function $S$ with the property that $L^*(t, x, v) \geq 0$ for all line elements in $(t, x, v) \in \mathscr{A}$ and for which one could find a continuous function $\psi(t, x)$, so that $L^*(t, x, \psi(t, x)) = 0$, i. e., $L^*(t, x, v)$ takes, for each $(t, x) \in \mathscr{R}$, its minimum value zero at $v = \dot{x} = \psi(t, x)$.

Then

$$\begin{aligned} I(x') - I(x) &= \int_{t_1}^{t_2} L(t, x'(t), \dot{x}'(t))\,\mathrm{d}t - \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t))\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} L^*(t, x'(t), \dot{x}'(t))\,\mathrm{d}t - \int_{t_1}^{t_2} L^*(t, x(t), \dot{x}(t))\,\mathrm{d}t \\ &\quad + S(t_1, x'(t_1)) - S(t_2, x'(t_2)) - S(t_1, x(t_1)) + S(t_2, x(t_2)) \\ &= \int_{t_1}^{t_2} L^*(t, x'(t), \dot{x}'(t))\,\mathrm{d}t - \int_{t_1}^{t_2} L^*(t, x(t), \dot{x}(t))\,\mathrm{d}t \geq 0\,. \qquad \square \end{aligned}$$

□

4th Stage: Fundamental equations of the calculus of variations. This immediately leads to Carathéodory's fundamental equations of the calculus of variations, here directly written for Lagrangian problems: Introducing the Lagrange function

$$M(t, x, \dot{x}, \mu) := L(t, x, \dot{x}) + \mu^\top G(t, x, \dot{x})$$

with the $p$-dimensional Lagrange multiplier $\mu$, the fundamental equations are

$$S_x = M_{\dot{x}}(t, x, \psi, \mu)\,, \tag{9}$$

$$S_t = M(t, x, \psi, \mu) - M_{\dot{x}}(t, x, \psi, \mu)\,\psi\,, \tag{10}$$

$$G(t, x, \psi) = 0\,. \tag{11}$$

These equations can already be found in Carathéodory's paper [6] of 1926, almost 30 years prior to Bellman's version of these equations. They constitute necessary conditions for an extremal of (1), (2).

**Remark 1.** In his short note of 1967 [18], seized again in 2001 [19], Leitmann describes a similar, though different, approach using a coordinate transformation to construct an equivalent variational problem, too, in which the minimizers are in one-to-one correspondence to the minimizers of the original problem. Carlson [11] and Carlson and Leitmann [12], [13] then showed that the equivalent problem takes a particularly simple form if the coordinate transformation is performed by means of a field $\mathscr{F} := \{x \in PC^1([t_1,t_2],\mathscr{R}^n) \mid x(t_1) = a, x(t_2) = b\}$ of extremals — here to be understood as solutions of the Euler-Lagrange equations —, so that the equivalent problem may be solved by inspection. Their most recent paper also considers differential constraints [14].

Wagener [25] finally showed the equivalence of the equivalences, i. e. Carathéodory's equivalence turns out to be a special case of Leitmann's equivalence. Leitmann's coordinate transformation is based on a diffeomorphism $\Xi$ defined by $\Xi(t,x) = (t,\xi(t,x))$ which gives rise to an bijective operator $\mathscr{X}\colon \mathscr{F} \to \mathscr{F}^*$, $x \mapsto y$, with $\mathscr{F}^* := \{y \in PC^1([t_1,t_2],\mathscr{R}^n) \mid y(t_1) = a^*, y(t_2) = b^*$ with $a^* = \xi(t_1,a)\,,\ b^* = \xi(t_2,b)\}$. Here the choice $\Xi = \mathrm{id}$ in Leitmann's equivalence approach yields Carathéodory's approach. If $\Xi$ can now be constructed in such a way that a regular field of extremals of the original problem is rectified to a field of extremals of the equivalent problem, Leitmann's equivalence of Carathéodory's fundamental equations (9)–(11) can be solved for certain classes of problems very easily. Hence the inverse $\Xi^{-1}$ must yield the rectification in this connection, i. e. $\xi$ must embed the minimizer $\bar{x}$ of the original problem in the form $\bar{x}(t) = \xi(t,\bar{y})$ with a constant minimizer $\bar{y}$ of the Leitmann equivalent problem. Note that solving the Hamilton-Jacobi equation (10), Carathéodory has to integrate $\dot{S}(t,x(t)) = L(t,x(t),\dot{x}(t))$ along extremals whereas, in Leitmann's approach, this equation reduces to $S_t^*(t,y) = \hat{L}(t,y,0)$ with $\hat{L}(t,y,\dot{y}) = L(t,\xi(t,y),\xi_t(t,y) + \xi_y(t,y)\,\dot{y})$.

### 2.3. **Second section of road: heading for Weierstrass' necessary condition.**

5th Stage: Necessary condition of Weierstrass. Replacing $\psi$ by $\dot{x}$ in the right hand sides of (9)–(11), Weierstrass' Excess Function for the Lagrange problem (1), (2) is obtained as

$$\mathscr{E}(t,x,\dot{x},x',\mu) = M(t,x,x',\mu) - M(t,x,\dot{x},\mu) - M_{\dot{x}}(t,x,\dot{x},\mu)\;(x'-\dot{x}) \tag{12}$$

with line elements $(t,x,\dot{x})$ and $(t,x,x')$ both satisfying the constraint (2). By a Taylor expansion, it can be easily seen that the validity of the Legendre-Clebsch condition in a certain neighborhood of the line element $(t,x,\dot{x})$ is a sufficient condition for the necessary condition of Weierstrass,

$$\mathscr{E}(t,x,\dot{x},x',\mu) \geq 0\,. \tag{13}$$

The Legendre-Clebsch condition can then be formulated as follows: The minimum of the quadratic form

$$Q = \xi^\top M_{\dot{x}\,\dot{x}}(t,x,\dot{x},\mu)\,\xi\,,$$

subject to the constraint

$$\frac{\partial G}{\partial \dot{x}}\,\xi = 0$$

on the sphere $\|\xi\|_2 = 1$, must be positive. This immediately implies

$$\begin{pmatrix} M_{\dot{x}\,\dot{x}} & G_{\dot{x}}^\top \\ G_{\dot{x}} & 0 \end{pmatrix} \quad \text{must be positive semi-definite}\,. \tag{14}$$

This result will play an important role when canonical coordinates are now introduced.

6th Stage: Canonical coordinates and Hamilton function. New variables are introduced by means of

$$y := M_{\dot{x}}^{\top}(t,x,\dot{x},\mu)\,, \tag{15}$$

$$z := G(t,x,\dot{x}) = M_{\mu}^{\top}(t,x,\dot{x},\mu)\,. \tag{16}$$

Because of (14), these equations can be solved for $\dot{x}$ and $\mu$ in a neighborhood of a *"minimal element"* $(t,x,\dot{x},\mu)$,[13]

$$\dot{x} = \Phi(t,x,y,z)\,, \tag{17}$$

$$\mu = X(t,x,y,z)\,. \tag{18}$$

Defining the Hamiltonian in canonical coordinates $(t,x,y,z)$ by

$$H(t,x,y,z) = -M(t,x,\Phi,X) + y^{\top}\,\Phi + z^{\top}\,X\,, \tag{19}$$

the function $H$ is at least twice continuously differentiable and there holds

$$H_t = -M_t\,,\quad H_x = -M_x\,,\quad H_y = \Phi^{\top}\,,\quad H_z = X^{\top}\,. \tag{20}$$

Letting $\mathscr{H}(t,x,y) = H(t,x,y,0)$, the first three equations of (20) remain valid for $\mathscr{H}$ instead of $H$. Alternatively, $\mathscr{H}$ can be obtained directly from $y = M_{\dot{x}}^{\top}(t,x,\dot{x},\mu)$ and $0 = G(t,x,\dot{x})$ because of (14) via the relations $\dot{x} = \phi(t,x,y)$ and $\mu = \chi(t,x,y)$,

$$\mathscr{H}(t,x,y) = -L(t,x,\phi(t,x,y)) + y^{\top}\,\phi(t,x,y)\,. \tag{21}$$

Note that $\phi$ is at least of class $C^1$ because $L \in C^2$, hence $\mathscr{H}$ is at least $C^1$, too. The first derivatives of $\mathscr{H}$ are, by means of the identities $y = L_{\dot{x}}^{\top}(t,x,\dot{x})^{\top}$,

$$\mathscr{H}_t(t,x,y) = -L_t(x,y,\phi)\,,\quad \mathscr{H}_x(t,x,y) = -L_x(t,x,\phi)\,,$$

$$\mathscr{H}_y(t,x,y) = \phi(t,x,y)^{\top}\,.$$

Therefore, $\mathscr{H}$ is even at least of class $C^2$. This Hamilton function can also serve to characterize the variational problem completely.

### 2.4. **Carathéodory's Missed Exit to Optimal Control.**

7th Stage: Carathéodory's closest approach to optimal control. In Carathéodory's book [7] of 1935, p. 352, results are presented that can be interpreted as introducing the distinction between state and control variables in the implicit system of differential equations (2). Using an appropriate numeration and partition $x = (x^{(1)}, x^{(2)})$, $x^{(1)} := (x_1,\ldots,x_p)$, $x^{(2)} := (x_{p+1},\ldots,x_n)$, Eq. (2) can be rewritten due to the rank condition (3):[14]

$$G(t,x,\dot{x}) = \dot{x}^{(1)} - \Psi(t,x,\dot{x}^{(2)}) = 0\,.$$

By the above equation, the Hamiltonian (21) can be easily rewritten as

$$\mathscr{H}(t,x,y) = -\bar{L}(t,x,\phi^{(2)}) + {y^{(1)}}^{\top}\,\phi^{(1)} + {y^{(2)}}^{\top}\,\phi^{(2)} \tag{22}$$

$$\text{with}\quad \bar{L}(t,x,\phi^{(2)}) := L(t,x,\Psi,\phi^{(2)})$$

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[13] Carathéodory has used only the term *extremal element* $(t,x,\dot{x},\mu)$ depending whether the matrix (14) is positive or negative semi-definite. For, there exists a $p$-parametric family of extremals that touches oneself at a line element $(t,x,\dot{x})$. However, there is only one extremal through a regular line element $(t,x,\dot{x})$.

[14] The original version is $\Gamma_{k'}(t,x_j,\dot{x}_j) = \dot{x}_{k'} - \Psi_{k'}(t,x_j,\dot{x}_{j''}) = 0$, where $k' = 1,\ldots,p$, $j = 1,\ldots,n$, $j'' = p+1,\ldots,n$. Note that Carathéodory used $\Gamma$ in his book of 1935 instead of $G$ which he used in his paper of 1926 and which we have inherit here.

FIGURE 2. Constantin Carathéodory in Göttingen (1904, the year of receiving his doctorate), his office in his home in Munich, and in Munich (1932, until which most of the material presented here was already developed)[15]

and $\dot{x}^{(1)} = \Psi(t, x, \phi^{(2)}) = \phi^{(1)}(t, x, y)$ and $\dot{x}^{(2)} = \phi^{(2)}(t, x, y)$. This is exactly the type of Hamiltonian known from optimal control theory. The canonical variable $y$ stands for the costate and $\dot{x}^{(2)}$ for the remaining freedom of the optimization problem (1), (2) later denoted by the control.

Nevertheless, the first formulation of a problem of the calculus of variations as an optimal control problem, which can be designated justifiably so, can be found in Hestenes' RAND Memorandum [15] of 1950.

8th Stage: Weierstrass' necessary condition in terms of the Hamiltonian. From Eqs. (13), (15), (16), (19), and (20) there follows Carathéodory's formulation of Weierstrass' necessary condition which can be interpreted as a precursor of the maximum principle

$$\mathscr{E} = \mathscr{H}(t, x, y) - \mathscr{H}(t, x, y') - \mathscr{H}_y(t, x, y')\,(y - y') \geq 0\,, \tag{23}$$

where $(t, x, y)$ and $(t, x, y')$ are the canonical coordinates of two line elements passing through the same point. This formula can already be found in Carathéodory's paper [6] of 1926.

3. **Side road to a maximum principle of Optimal Control Theory.** In Pesch, Bulirsch (1994), a proof for the maximum principle was given for an optimal control problem of type

$$\int_{t_1}^{t_2} L(t, z, u)\,\mathrm{d}t \overset{!}{=} \min \quad \text{subject to} \quad \dot{z} = g(t, z, u)$$

starting with Carathéodory's representation of Weierstrass' necessary conditions (23) in terms of a Hamiltonian.

In the following we pursue a different way leading to the maximum principle more directly, still under the too strong assumptions of the calculus of variations. Herewith, we continue the tongue-in-cheek story on 300 years of Optimal Control by Sussmann and Willems (1997) by adding a little new aspect.

[15] All pictures by courtesy of Mrs. Despina Carathéodory-Rodopoulou, daughter of Carathéodory. See: Δ. Καραθεοδωρή-Ροδοπούλου, Δ. Βλαχοστεργίου-Βασβατέκη: Κωνσταντίνος Καραθεοδωρή: Ο σοφός ᾿Ελλην του Μονάχου, Εκδόσεις Κάκτος, Athens, 2001.

Picking up the fact that $\dot{x} = v(t,x)$ minimizes $v \mapsto L_v^*(t,x,v)$, we are led by (6) to the costate $p = L_v^\top(t,x,\dot{x})$ [as in (15), now using the traditional notation] and the Hamiltonian $H$,

$$H(t,x,p) = \min_{\dot{x}} \{L(t,x,\dot{x}) + p^\top \dot{x}\}\,.$$

Then Carathéodory's fundamental equations read as follows

$$p = -S_x^\top(t,x)\,, \quad S_t = H(t,x,S_x^\top)\,;$$

compare Wagener [25]. This is the standard form of the Hamiltonian in the context of the calculus of variations leading to the Hamilton-Jacobi equation.

Following however Sussmann and Willems (1997) we are led to the now maximizing Hamiltonian (since we are aiming to a maximum principle), also denoted by $H$,

$$H(t,x,u,p) = -L(t,x,u) + p^\top u$$

with $p = L_u^\top(t,x,u)$ defined accordingly and the traditional notation for the degree of freedom, the control $\dot{x} = u$, when we restrict ourselves, for the sake of simplicity, to the most simplest case of differential constraints.

It is then obvious that $H_p^\top = u$ as long as the curve $x$ satisfies

$$\dot{x}(t) = H_p^\top(t,x(t),\dot{x}(t),p(t))\,. \tag{24}$$

By means of the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} L_u(t,x,\dot{x}) - L_x(t,x,\dot{x}) = 0$$

and because of $H_x = -L_x$, we obtain

$$\dot{p}(t) = -H_x^\top(t,x,\dot{x},p(t))\,. \tag{25}$$

Furthermore, we see $H_u^\top = -L_u^\top + p = 0$. Since the Hamiltonian $H(t,x,u,p)$ is equal to $-L(t,x,u)$ plus a linear function in $u$, the strong Legendre-Clebsch condition for now maximizing the functional (1) is equivalent to $H_{uu} < 0$. Hence $H$ must have a maximum with respect to $u$ along a curve $(t,x(t),p(t))$ defined by the above canonical equations (24), (25).

If $L$ depends linearly on $u$, the maximization of $H$ makes sense only in the case of a constraint on the control $u$ in form of a closed convex set $U_{\mathrm{ad}}$ of admissible controls, which would immediately yield the variational inequality

$$H_u(t,x,\bar{u},p)\,(u-\bar{u}) \le 0 \quad \forall u \in U_{\mathrm{ad}} \tag{26}$$

along a candidate optimal trajectory $x(t)$, $p(t)$ satisfying the canonical equations (24), (25) with $\bar{u}$ denoting the maximizer. That is the maximum principle in its known modern form.

A missed exit from the royal road of the calculus of variations to the maximum principle of optimal control? Not at all! However, it could have been at least a first indication of a new field of mathematics looming on the horizon.

## 4. Conclusions.

With Carathéodory's own words: *I will be glad if I have succeeded in impressing the idea that it is not only pleasant and entertaining to read at times the works of the old mathematical authors, but that this may occasionally be of use for the actual advancement of science. [...] We have seen that even under conditions which seem most favorable very important results can be discarded for a long time and whirled away from the main stream which is carrying the vessel science. [...] It may happen that the work of most celebrated men may be overlooked. If their*

FIGURE 3. Professor Dr. Dr.h.c.mult. George Leitmann together with Dr. Heinz Fischer, The Federal President of the Republic of Austria, in the Vienna Hofburg, May 2010, on the occasion of George Leitmann's 85th birthday[16]

*ideas are too far in advance of their time, and if the general public is not prepared to accept them, these ideas may sleep for centuries on the shelves of our libraries. [...] But I can imagine that the greater part of them is still sleeping and is awaiting the arrival of the prince charming who will take them home.*[16]

**Acknowledgments.** The author would like to express his sincere appreciation to Professor George Leitmann, one of the scientific heroes in Optimal Control Theory and Differential Games for his continuous influence through his entire scientific career. Professor Leitmann has always acted as a role model for young researchers such as this author ever since we first met. As a German, the author would particularly like to thank George Leitmann for his friendship over so many years and for all the outstanding services he has rendered to Germany despite the horrible times through which he was forced to go as a 15 year old boy, expelled from his childhood in Vienna, on the run from the pandemonium of darkness that has strangled millions. George Leitmann is one of the most influential, productive, and simultaneously decent and likable colleagues the author has ever met.

[16]on Aug. 31, 1936, at the meeting of the Mathematical Association of America in Cambridge, Mass., during the tercentenary celebration of Harvard University; see Carathéodory [8]

[16]Picture by courtesy of Professor Vladimir Veliov, Institute of Mathematical Methods in Economics, Vienna University of Technology

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Received xxxx 20xx; revised xxxx 20xx.

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