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Maximal monotonicity of bifunctions

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For each monotone bifunction F defined on a subset C of a Banach space, an associated monotone operator A^F can be defined. The bifunction F is called maximal monotone if A^F is maximal monotone. We find conditions for a bifunction to be maximal monotone and show the relation to the existence of solutions of an equilibrium problem. Also, we establish some properties of the domain C when F is maximal monotone. Finally, we define and study cyclically monotone bifunctions.

Keywords: Equilibrium problem; Monotone bifunctions; Maximal monotone operators; Cyclic monotonicity

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1 Introduction

Given a nonempty subset C of a Banach space X, by the term "bifunction" we understand any function $F : C \times C \to \mathbb{R}$ such that $F(x, x) = 0$ for all $x \in C$. A bifunction F is called monotone if

$$\forall x, y \in C, \quad F(x, y) + F(y, x) \leq 0.$$

Monotone bifunctions were systematically studied in the seminal paper by Blum and Oettli [1] in relation with the following equilibrium problem: find $x_0 \in C$ such that

$$\forall y \in C, \quad F(x_0, y) \geq 0.$$

The equilibrium problem unifies a lot of different problems in Optimization and Nonlinear Analysis (see [1] for a detailed presentation) and for this reason the existence of solutions for equilibrium problems was studied in a large

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number of papers. Recently, there were many papers devoted to algorithms for finding such solutions, see, e.g., [2–6].

In the present paper we study monotone bifunctions from another viewpoint. To each bifunction F corresponds a monotone operator A^F defined by $A^F(x) = \partial F(x,\cdot)(x)$. A monotone bifunction F will be called maximal monotone if A^F is maximal monotone. Our aim is to study some properties of maximal monotone bifunctions, with regard to their domain of definition, equilibrium problems, cyclic monotonicity etc.

In Section 2 we define maximal monotonicity of bifunctions, and show the relation between this notion and a notion of resolvent for bifunctions. In Section 3 we give several kinds of criteria that guarantee maximal monotonicity of bifunctions. In Section 4 we investigate some consequences of maximal monotonicity regarding the set C. In particular, we give general conditions that guarantee that $\overline{C}$ is convex or that $\operatorname{int} C = \operatorname{int} D(A^F)$. Finally in Section 5 we introduce and study cyclically monotone bifunctions. It is shown that a bifunction F is cyclically monotone if and only if there exists a function $f : C \to \mathbb{R}$ such that $F(x,y) \le f(y) - f(x)$ for all $x, y \in C$. If in particular F is maximal monotone and $\operatorname{int} C \neq \emptyset$, then f is uniquely defined up to a constant and convex on $\operatorname{int} C$.

As the results of the paper show, maximal monotone bifunctions have some properties that can be expected from similar properties of maximal monotone operators, but we provide several examples to show that other properties that seem plausible are not, in fact, true.

In the following, X will denote a reflexive Banach space X equipped with a norm such that both X and X^* are strictly convex (by Troyanski's theorem, any reflexive Banach space can be renormed with an equivalent norm so that both X and X^* are locally uniformly convex, and this implies strict convexity; see [7]). In this case the duality map

$$\mathcal{J}(x) = \left\{ x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2 \right\}$$

is single-valued, surjective, odd, strictly monotone and demicontinuous (i.e., continuous when X and X^* are equipped, respectively, with the strong and the weak topology) [7]. We denote by B the open unit ball of X. For any multivalued operator $A : X \to 2^{X^*}$ we denote by $D(A)$ its domain and by $\operatorname{gr} A$ its graph. Given any function $f : X \to \mathbb{R} \cup \{+\infty\}$ (not necessarily convex) with domain $\operatorname{dom} f$, its (Fenchel Moreau) subdifferential is defined as the multivalued operator ∂f where

$$\partial f(x) = \begin{cases} \{x^* \in X^* : \forall y \in X,\ f(y) - f(x) \ge \langle x^*, y - x \rangle\}, & x \in \operatorname{dom} f \\ \emptyset, & x \notin \operatorname{dom} f \end{cases}$$

A function $f : C \to \mathbb{R}$ is called upper hemicontinuous if its restriction on each line segment of C is upper semicontinuous. An operator $A : X \to 2^{X^*}$ is called upper hemicontinuous if its restriction on each line segment of $D(A)$ is upper semicontinuous (as a multivalued map) with respect to the weak* topology in X^*.

2 Maximal monotonicity of bifunctions

Let $F : C \times C \to \mathbb{R}$ be a monotone bifunction. Following [8], we define the operator $A^F : X \to 2^{X^*}$ as follows:

$$A^F(x) = \begin{cases} \{x^* \in X^* : \forall y \in C,\, F(x,y) \geq \langle x^*, y-x\rangle\}, \text{ if } x \in C \\ \emptyset, \text{ if } x \in X \backslash C. \end{cases}$$

In fact, if we define an extension $\hat{F}$ of F on $C \times X$ by setting $\hat{F}(x,y) = +\infty$ for $x \in C$, $y \in X \backslash C$, then we see that $A^F(x) = \partial \hat{F}(x,\cdot)(x)$ for all $x \in C$. In the special case where $F(x,y) = \phi(y) - \phi(x)$ where ϕ is any function with $\operatorname{dom} \phi = C$, one has $A^F = \partial \phi$. However, in general A^F is not defined as the subdifferential operator of a unique function since for each $x \in C$, $\hat{F}(x,\cdot)$ is a different function. Nevertheless, one can show that A^F is monotone with the same proof as for the subdifferential of a unique function [8]. Note that just as with subdifferentials, the values of A^F are convex and closed.

Definition 2.1 A monotone bifunction F is called maximal monotone if A^F is maximal monotone.

The previous definition of maximal monotonicity for bifunctions differs from the one initially given by Blum and Oettli [1]. We recall that a monotone bifunction $F : C \times C \to \mathbb{R}$ is called maximal monotone in the sense of Blum and Oettli [1] (briefly, BO-maximal monotone) if for every $(x, x^*) \in C \times X^*$ the following implication holds:

$$\forall y \in C,\, F(y,x) + \langle x^*, y-x\rangle \leq 0 \;\Rightarrow\; \forall y \in C,\, F(x,y) \geq \langle x^*, y-x\rangle\,. \quad (1)$$

We provide an example of a monotone bifunction F which is BO-maximal monotone and such that $D(A^F) = \emptyset$.

Example 2.2 Let $f : X \to \mathbb{R}$ be any odd function which is not a member of X^*. Define $F : X \times X \to \mathbb{R}$ by $F(x,y) = f(y-x)$. Then F is monotone. It is obvious that F is BO-maximal monotone (in fact, there is no $(x, x^*) \in X \times X^*$ that satisfies either side of implication (1)). Assume that $x^* \in A^F(x)$ for some

$x \in X$. Then for all $y \in X$,

$$f(y) = F(x, y + x) \geq \langle x^*, y \rangle .$$

Applying the above relation to $-y$, we deduce that $f(y) = \langle x^*, y \rangle$ thus contradicting our assumption that f is not a continuous linear functional. For instance, if $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $F(x, y) = (y - x)^3$, then $D(A^F) = \emptyset$.

See [8], as well as Proposition 3.3 for the relation between the two definitions of maximal monotonicity.

For each $\lambda > 0$, the obvious identity $A^{\lambda F} = \lambda A^F$ implies that F is maximal monotone if and only if λF is maximal monotone. Also, if F is maximal monotone and $G : C \times C \to \mathbb{R}$ is a monotone bifunction and $F \leq G$, then G is also maximal monotone.

Given any monotone operator $T : X \to 2^{X^*}$, it is easy to check that for $x, y \in D(T)$,

$$\sup_{x^* \in T(x)} \langle x^*, y - x \rangle + \sup_{y^* \in T(y)} \langle y^*, x - y \rangle \leq 0$$

and in particular $\sup_{x^* \in T(x)} \langle x^*, y - x \rangle \in \mathbb{R}$. Thus the bifunction $G_T : D(T) \times D(T) \to \mathbb{R}$ defined by

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \tag{2}$$

is monotone and satisfies $G_T(x, x) = 0$, $\forall x \in D(T)$. We have the following easy proposition, which shows some consistency of our definitions.

Proposition 2.3 *If the operator T is maximal monotone then G_T is maximal monotone. In fact, $A^{G_T} = T$.*

Proof Set $\widetilde{T} := A^{G_T}$. Assume that T is maximal monotone. If $x \in D(T)$, $x^* \in T(x)$ then for each $y \in D(T)$, $G_T(x, y) \geq \langle x^*, y - x \rangle$ by the definition of G_T. This means that $x^* \in \widetilde{T}(x)$, hence $\widetilde{T}$ is an extension of T, i.e., $\widetilde{T}$ is maximal monotone. This means that G_T is maximal monotone. □

Note that the converse does not hold in general, as it is possible to have $G_T = G_S$ for two monotone operators T and S, while $T \neq S$. For instance if T is maximal monotone and S is any operator different from T such that $\overline{\mathrm{co}} S(x) = T(x)$ for all $x \in X$, then $G_S = G_T$ hence G_S is maximal monotone, while S is not. Another example: T is the operator with domain $D(T) = [0, +\infty)$ defined by $T(x) = \{0\}$, $\forall x \in [0, +\infty)$. Then $G_T : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is given by $G_T(x, y) = 0$ if $x \in [0, +\infty)$ and A^{G_T} is given by $A^{G_T}(x) = \{0\}$ if $x \in (0, +\infty)$

and $A^{G_T}(0) = (-\infty, 0]$. Thus, A^{G_T} and G_T are maximal while T is not. Note that T has closed convex values. On the other hand, it is easy to see that the converse of Proposition 2.3 holds under assumptions that exclude the above examples:

PROPOSITION 2.4 *Assume that T is monotone, has closed convex values, and $D(T) = X$. If G_T is maximal monotone, then T is maximal monotone.*

Proof By assumption, A^{G_T} is maximal monotone. For each $x \in X$ and $z^* \in A^{G_T}(x)$ one has

$$\forall y \in X, \quad \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq \langle z^*, y - x \rangle .$$

By an immediate consequence of the separation theorem, the above inequality implies that $z^* \in T(x)$. Hence $A^{G_T}(x) \subseteq T(x)$ and by maximality of A^{G_T} we deduce that T is maximal monotone. $\square$

Given a maximal monotone bifunction F, one can construct A^F and the monotone bifunction $G := G_{A^F}$. One has $G(x, y) \leq F(x, y)$ for all $x, y \in D(A^F)$. According to Proposition 2.3, $A^F = A^G$. As the following example shows, in general F and G are not equal, thus showing that the correspondence $F \to A^F$ is not one-to-one.

Example 2.5 Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $F(x, y) = y^2 - x^2$. Then $A^F(x) = \{2x\}$, hence F is maximal monotone. One has $G_{A^F}(x, y) = 2x(y - x)$. Note that F and $G := G_{A^F}$ are both maximal monotone, $A^F = A^G$ but $G(x, y) < F(x, y)$ unless $x = y$.

The idea and a proof of the following proposition is in essence contained in [8]. We include a short proof for the sake of completeness.

PROPOSITION 2.6 *A monotone bifunction F is maximal monotone if and only if for each $\lambda > 0$ (equivalently, for some $\lambda > 0$) and each $x \in X$ there exists $x_\lambda \in C$ such that*

$$\forall y \in C, \quad \lambda F(x_\lambda, y) + \langle \mathcal{J}(x_\lambda - x), y - x_\lambda \rangle \geq 0. \tag{3}$$

This element x_λ is uniquely defined.

Proof It is known that a monotone operator T is maximal monotone if and only if for each $x \in X$ there exists $x' \in X$ such that $0 \in \mathcal{J}(x' - x) + T(x')$; this element x' is unique in case $\mathcal{J}$ is strictly monotone [7, pg 324], which is true under our assumption that X and X^* are strictly convex. Applying this to λA^F for some $\lambda > 0$, we get that A^F is maximal monotone if and only if for each $x \in X$ there exists a unique $x_\lambda \in X$ such that $\frac{-1}{\lambda}\mathcal{J}(x_\lambda - x) \in A^F(x_\lambda)$. This is obviously equaivalent to (3). □

Note that (3) implies that for every $x \in X$, $x_\lambda \in D(A^F)$. The operator $J_\lambda^F : X \to D(A^F)$ defined by $J_\lambda^F(x) = x_\lambda$ is called the resolvent of F. Hence the resolvent satisfies the following equation:

$$\forall x \in X, \forall y \in C, \quad \lambda F(J_\lambda^F(x), y) + \left\langle \mathcal{J}\left(J_\lambda^F(x) - x\right), y - J_\lambda^F(x)\right\rangle \geq 0. \tag{4}$$

3 Criteria for maximal monotonicity

We now give a condition that guarantees maximality for F. See also [1] and [8].

Proposition 3.1 *Let $C \subseteq X$ be nonempty, closed and convex. If F is monotone, $F(\cdot, y)$ is upper hemicontinuous (i.e., upper semicontinuous on line segments) for all $y \in C$ and $F(x, \cdot)$ is convex and l.s.c. for all $x \in C$, then F is maximal monotone.*

Proof For any $\lambda > 0$ and $x \in X$ define the bifunction $F^{\lambda,x} : C \times C \to \mathbb{R}$ by

$$F^{\lambda,x}(z, y) = \lambda F(z, y) + \langle \mathcal{J}(z - x), y - z\rangle. \tag{5}$$

This bifunction is monotone since

$$\begin{aligned} F^{\lambda,x}(z, y) + F^{\lambda,x}(y, z) &= \lambda(F(z, y) + F(y, z)) + \langle \mathcal{J}(z - x) - \mathcal{J}(y - x), y - z\rangle \\ &\leq -\langle \mathcal{J}(z - x) - \mathcal{J}(y - x), (z - x) - (y - x)\rangle \qquad (6) \\ &\leq -(\|z - x\| - \|y - x\|)^2. \end{aligned}$$

Also, it is evident that $F^{\lambda,x}(z, \cdot)$ is convex and l.s.c. for all $z \in C$. In addition, $F^{\lambda,x}(\cdot, y)$ is upper hemicontinuous for all $y \in C$ since $\mathcal{J}$ is demicontinuous. Finally choose any $z_0 \in C$. Since $F^{\lambda,x}(z_0, \cdot)$ is convex and l.s.c., it has an affine minorant; i.e., there exist $z^* \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$\forall y \in C, \quad F^{\lambda,x}(z_0, y) \geq \langle z^*, y\rangle + \alpha. \tag{7}$$

Combining (7) with (6) we deduce that whenever C is unbounded, one has

$$\lim_{\|y\| \to +\infty} F^{\lambda,x}(y, z_0) = -\infty.$$

Hence, all assumptions of Theorem 1 in [1] are satisfied and the equilibrium problem

$$\forall y \in C, \quad F^{\lambda,x}(z,y) \geq 0$$

has a solution. The result follows from Proposition 2.6. □

The above proposition can be generalized as follows:

PROPOSITION 3.2 *Let $C \subseteq X$ be nonempty, closed and convex. Assume that $F(\cdot,y)$ is upper hemicontinuous for all $y \in C$ and $F(x,\cdot)$ is convex and l.s.c. for all $x \in C$. Further, let $G : C \times C \to \mathbb{R}$ be another monotone bifunction such that $G(\cdot,y)$ is u.s.c. for all $y \in C$, $G(x,\cdot)$ is convex for all $x \in C$, and for all $y \in C$, $\limsup_{\|x\|\to+\infty} \frac{G(x,y)}{\|x\|} < +\infty$. Then $F+G$ is maximal monotone.*

Proof Define $F^{\lambda,x}$ as in the proof of the previous proposition. Then using (6), (7), and our assumption on G we deduce that there exists $M \in \mathbb{R}$ such that

$$F^{\lambda,x}(y,z_0) + \lambda G(y,z_0) \leq -\left(\langle z^*, y\rangle + \alpha\right) - \left(\|z_0 - x\| - \|y - x\|\right)^2 + \lambda M \|y\|$$

for sufficiently large $\|y\|$. Hence

$$\lim_{\|y\|\to+\infty} \Big(F^{\lambda,x}(y,z_0) + \lambda G(y,z_0)\Big) = -\infty$$

We deduce again that all assumptions of Theorem 1 in [1] are satisfied and the equilibrium problem

$$\forall y \in C, \quad F^{\lambda,x}(z,y) + \lambda G(z,y) \geq 0 \tag{8}$$

has a solution. As before, the result follows from Proposition 2.6. □

The following proposition was also shown in [8] in the special case $G = 0$. Here we provide a shorter proof for the more general case.

PROPOSITION 3.3 *Assume that C is nonempty, closed and convex and F, G are monotone bifunctions such that: F is BO-maximal monotone, $F(x,\cdot)$ is l.s.c. and convex, $G(\cdot,y)$ is upper semicontinuous, $G(x,\cdot)$ is convex and $\limsup_{\|x\|\to+\infty} \frac{G(x,y)}{\|x\|} < +\infty$. Then $F+G$ is maximal monotone.*

Proof Choose $z_0 \in C$. Since $F(z_0,\cdot)$ is convex and l.s.c., there exist $z^* \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$\forall y \in C, \quad F(z_0,y) \geq \langle z^*, y\rangle + \alpha.$$

It is clear from our assumption on G that for each $x \in X$,

$$\lim_{\|y\|\to+\infty} \left(\langle z^*, y\rangle + \alpha - G(y, z_0) - \langle \mathcal{J}(y-x), z_0 - x\rangle + \|y-x\|^2\right) = +\infty.$$

Hence for $\|y\|$ large enough,

$$\begin{aligned} F(z_0, y) &\geq \langle z^*, y\rangle + \alpha > G(y, z_0) + \langle \mathcal{J}(y-x), z_0 - x\rangle - \|y-x\|^2 \\ &= G(y, z_0) + \langle \mathcal{J}(y-x), z_0 - y\rangle . \end{aligned}$$

If we set $g(z, y) = F(z, y)$ and $h(y, z) = G(y, z) + \langle \mathcal{J}(y-x), z-y\rangle$, we see that all assumptions of Theorem 1A in [1] are satisfied. Hence the equilibrium problem (8) has a solution, and the result follows again from Proposition 2.6. □

The above propositions provide a large class of maximal monotone bifunctions. However, as the following example shows, this class is not exhaustive:

Example 3.4 Let $F : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}$ be given by $F(x, y) = x(x^2 - y^2)$. Then F is monotone. A^F can be readily calculated:

$$A^F(x) = \begin{cases} \{-2x^2\}, \text{ if } x < 0 \\ \mathbb{R}^+, \text{ if } x = 0. \end{cases}$$

It is obvious that A^F is maximal monotone. Now define $F_1 : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}$ by

$$F_1(x, y) = \begin{cases} F(x, y), & y < \frac{x}{2} \\ -\frac{5x^2}{2} y + 2x^3, & \frac{x}{2} \leq y \leq 0. \end{cases}$$

Note that for $(x, y) \in \mathbb{R}^- \times \mathbb{R}^-$, $-2x^2(y - x) \leq F_1(x, y) \leq F(x, y)$ (in fact, we have only to check the inequalities when $\frac{x}{2} < y \leq 0$; in this case, $F_1(x, y) \leq F(x, y)$ is equivalent to $\frac{y^2}{x^2} - \frac{5}{2}\frac{y}{x} + 1 \geq 0$, which is true for $0 \leq \frac{y}{x} \leq \frac{1}{2}$). It follows that F_1 is monotone and $-2x^2 \in A^{F_1}(x)$. Thus, F_1 is maximal monotone. Finally, $F_1(x, \cdot)$ is continuous but is not convex since $F_1(x, \frac{x}{2}) = \frac{3}{4}x^3 > x^3 = \frac{1}{2}F_1(x, 0) + \frac{1}{2}F_1(x, x)$.

Whenever F is defined on the whole space, we have another criterion for maximality:

Proposition 3.5 *Assume that $F : X \times X \to \mathbb{R}$ is upper hemicontinuous with respect to the first variable, monotone, and $D(A^F) = X$. Then F is maximal monotone.*

Proof Given $x \in X$, consider any straight line l through x, any sequence $\{x_n\} \subseteq l$ converging to x and any $x_n^* \in A^F(x_n)$. Since A^F is monotone and defined everywhere, it is locally bounded, hence there exists a subsequence $x_{n_i}^*$ weakly converging to some $x^* \in X$. Then

$$\forall y \in X, \quad F(x,y) \geq \limsup_{i\to+\infty} F(x_{n_i},y) \geq \limsup_{i\to+\infty} \left\langle x_{n_i}^*, y - x_{n_i}\right\rangle = \langle x^*, y-x\rangle .$$

It follows that $x^* \in A^F(x)$. This means that for every straight line l, $A^F : l \to 2^{X^*}$ is u.s.c. (see for instance Prop. 2.19, pg 41 of [7]) i.e., A^F is upper hemicontinuous. Since A^F has weakly closed and convex values, it follows from Theorem 1.33, pg 309 of [7] that A^F is maximal monotone. □

4 Properties of maximal monotone bifunctions

PROPOSITION 4.1 *Let $F : C \times C \to \mathbb{R}$ be maximal monotone. Assume that for every $x \in C$ and any converging sequence $\{x_n\} \subseteq C$, the sequence $\{F(x,x_n)\}$ is bounded from below. Then $C \subseteq \overline{D(A^F)}$. In particular, $\overline{C}$ is convex.*

Proof Using Troyanski's theorem [7, page 30] we can renorm X so that both X and X^* are locally uniformly convex. For each $x \in C$, set $x_\lambda = J_\lambda^F(x)$. Then $-\lambda^{-1}\mathcal{J}(x_\lambda - x) \in A^F x_\lambda$, hence for all $(y,y^*) \in \operatorname{gr} A^F$ we have

$$\left\langle y^* + \frac{1}{\lambda}\mathcal{J}(x_\lambda - x), y - x_\lambda \right\rangle \geq 0. \tag{9}$$

We deduce that

$$\|x_\lambda - x\|^2 \leq \lambda \langle y^*, y-x\rangle + \lambda \langle y^*, x - x_\lambda\rangle + \langle \mathcal{J}(x_\lambda - x), y - x\rangle . \tag{10}$$

It follows that for every sequence $\lambda_n \to 0^+$, x_{λ_n} is bounded. Thus there exists a subsequence (which we denote again by λ_n) such that x_{λ_n} is weakly converging to some $\overline{x}$. Since the space X is reflexive and A^F is maximal, we know that $\overline{D(A^F)}$ is closed and convex, hence weakly closed, so $\overline{x} \in \overline{D(A^F)}$.

We consider two cases. If some subsequence of x_{λ_n} strongly converges to x, then $x = \overline{x} \in \overline{D(A^F)}$ and we are finished. Otherwise, using (10) we get

$$\|x_{\lambda_n} - x\|^2 \leq \lambda_n \|y^*\| \|y-x\| + \lambda_n \|y^*\| \|x - x_{\lambda_n}\| + \|x_{\lambda_n} - x\| \|y - x\|$$

hence

$$\|x_{\lambda_n} - x\| \le \frac{\lambda_n \|y^*\| \|y - x\|}{\|x_{\lambda_n} - x\|} + \lambda_n \|y^*\| + \|y - x\| . \tag{11}$$

By taking the limit we obtain

$$\|\overline{x} - x\| \le \liminf_{n\to+\infty} \|x_{\lambda_n} - x\| \le \|y - x\| , \quad \forall y \in D(A^F)$$

$$\Rightarrow \|\overline{x} - x\| \le \liminf_{n\to+\infty} \|x_{\lambda_n} - x\| \le \|y - x\| , \quad \forall y \in \overline{D(A^F)}$$

Taking $y = \overline{x}$ we deduce that $\|\overline{x} - x\| = \liminf_{n\to+\infty} \|x_{\lambda_n} - x\|$. Thus, taking a subsequence if necessary, we can assume that $\|\overline{x} - x\| = \lim_{n\to+\infty} \|x_{\lambda_n} - x\|$. Since X is locally uniformly convex, it has the Kadec-Klee property [7], hence we deduce that $x_{\lambda_n} \to \overline{x}$ strongly. Using (3) we infer

$$\|x - x_{\lambda n}\|^2 \le \lambda_n F(x_{\lambda_n}, x) \le -\lambda_n F(x, x_{\lambda_n}).$$

By our assumption on F we obtain $\|x - \overline{x}\|^2 \le 0$ hence $x = \overline{x}$. This contradicts our assumption that no subsequence of x_{λ_n} strongly converges to x. Since $D(A^F) \subseteq C \subseteq \overline{D(A^F)}$ and $\overline{D(A^F)}$ is convex, we infer that $\overline{C} = \overline{D(A^F)}$ is convex. □

Corollary 4.2 *Assume that C is closed, F is maximal monotone, and that for every $x \in C$, $F(x, \cdot)$ is l.s.c.. Then C is convex and $C = \overline{D(A^F)}$.*

Proof Trivial application of the proposition. □

Corollary 4.3 *Assume that C is convex, F is maximal monotone, and that for every $x \in C$, $F(x, \cdot)$ is convex and l.s.c. at some point $y_0 \in C$. Then $C \subseteq \overline{D(A^F)}$.*

Proof Given $x \in C$, set $f(y) = F(x, y)$ for $y \in C$ and $f(y) = +\infty$ for $y \in X\backslash C$. Then f is convex, and also l.s.c. at y_0. It follows that its biconjugate f^{**} is proper, since $f^{**}(y_0) = f(y_0)$. Hence, f^{**} (and also f) are bounded below by a continuous affine function. Thus all assumptions of the previous proposition are satisfied. □

Proposition 4.4 *Assume that C is convex, and that $F(x, \cdot)$ is convex and l.s.c. for every $x \in C$. Then* $\operatorname{int} C = \operatorname{int} D(A^F)$.

Proof The assertion is obvious whenever $\operatorname{int} C = \emptyset$. Assume that $\operatorname{int} C \ne \emptyset$. For each $x_0 \in \operatorname{int} C$, the convex function $F(x_0, \cdot)$ is continuous at x_0, hence $\partial F(x_0, \cdot)(x_0) \ne \emptyset$. This means that $A^F(x_0) \ne \emptyset$, hence $\operatorname{int} C \subseteq D(A^F)$. □

Definition 4.5 A bifunction F is called locally bounded at a point $x_0 \in X$ if there exists a neighborhood V of x_0 and $k \in \mathbb{R}$ such that $F(x,y) \leq k$ for all $x, y \in V \cap C$.

We obtain the following version of Proposition 4.1 using local boundedness:

Proposition 4.6 *Assume that C is convex and F is maximal monotone and locally bounded at every point of $\overline{C}$. Then $C \subseteq \overline{D(A^F)}$.*

Proof As in the proof of Proposition 4.1 we renorm X, and then set for every $x \in C$, $x_\lambda = J_\lambda^F(x)$ and choose a sequence $\lambda_n \to 0^+$. We obtain again that either there exists a subsequence of x_{λ_n} strongly converging to x, in which case $x \in \overline{D(A^F)}$, or there exists a subsequence, which we denote again by x_{λ_n}, strongly converging to some element $\bar{x} \in \overline{C}$. Since F is locally bounded at $\bar{x}$, there exists a neighborhood $V = \bar{x} + 2\varepsilon B$ of $\bar{x}$ and $k \in \mathbb{R}$ such that for every $z, z' \in V \cap C$ one has $F(z, z') \leq k$. Assume that $x \neq \bar{x}$. Choose $t \in (0,1)$ such that $z := \bar{x} + t\,(x - \bar{x}) \in \bar{x} + \varepsilon B$. Let n be large enough so that $x_{\lambda_n} \in \bar{x} + \varepsilon B$. Set $z_{\lambda_n} = x_{\lambda_n} + t\,(x - x_{\lambda_n})$. Then

$$\|z - z_{\lambda_n}\| = \|(1-t)(\bar{x} - x_{\lambda_n})\| < \varepsilon.$$

Hence $\|z_{\lambda_n} - \bar{x}\| < 2\varepsilon$. Since C is convex, $z_{\lambda_n} \in C$. Setting $y = z_{\lambda_n}$ in (3) we obtain

$$\langle \mathcal{J}(x - x_{\lambda_n}), z_{\lambda_n} - x_{\lambda_n} \rangle \leq \lambda_n F(x_{\lambda_n}, z_{\lambda_n}) \leq \lambda_n k.$$

Taking the limit as $n \to +\infty$ we observe that $z_{\lambda_n} - x_{\lambda_n} \to z - \bar{x}$, hence demicontinuity of $\mathcal{J}$ implies $\langle \mathcal{J}(x - \bar{x}), z - \bar{x} \rangle \leq 0$. Since $z - \bar{x} = t(x - \bar{x})$ we obtain $\|x - \bar{x}\|^2 \leq 0$ which contradicts the assumption $x \neq \bar{x}$. Hence $x = \bar{x} \in \overline{D(A^F)}$. □

Whenever F is locally bounded on $\operatorname{int} C$, we can deduce that $\operatorname{int} C = \operatorname{int} D(A^F)$ without assuming that $F(x, \cdot)$ is convex:

Proposition 4.7 *Assume that F is maximal monotone and locally bounded on $\operatorname{int} C$ and that $C \subseteq \overline{D(A^F)}$. Then $\operatorname{int} C = \operatorname{int} D(A^F)$.*

Proof Given $x_0 \in \operatorname{int} C$, let $k \in \mathbb{R}$ and $\varepsilon > 0$ be such that $x_0 + 2\varepsilon B \subseteq C$ and $F(x,y) \leq k$ for all $x, y \in x_0 + 2\varepsilon B$. Then $x_0 \in \overline{D(A^F)}$ since $C \subseteq \overline{D(A^F)}$. For any $x \in D(A^F) \cap (x_0 + \varepsilon B)$, all $x^* \in A^F(x)$ and all $v \in \varepsilon B$ we have

$$\langle x^*, v \rangle = \langle x^*, x + v - x \rangle \leq F(x, x+v) \leq k$$

since $x + v \in x_0 + 2\varepsilon B$. Hence for every $x \in D(A^F) \cap (x_0 + \varepsilon B)$, $\sup_{x^* \in A^F(x)} \|x^*\| \leq k/\varepsilon$. It follows that A^F is locally bounded at x_0. By the

Libor-Veselý Theorem [9], $x_0 \in \operatorname{int} D(A^F)$. Hence, $\operatorname{int} C = \operatorname{int} D(A^F)$. $\square$

If $F : C \times C \to \mathbb{R}$ is a bifunction and $x_0 \in C$ is a solution of the equilibrium problem for F on C, i.e.,

$$\forall y \in C : F(x_0, y) \geq 0$$

then it is obvious that $0 \in A^F(x_0)$, i.e., x_0 is not just a solution for the variational inequality problem for A^F on C, but also a zero of A^F. However, usually we want to solve the equilibrium problem on a closed convex subset K of C. If we denote by F_K the restriction of F on K, then in general $A^{F_k} \neq A^F$ on K. For instance, if $C = \mathbb{R}$, $F(x, y) = x(y - x)$, $A^F(x) = \{x\}$ and we choose $K = [1, 2]$, then

$$A^{F_k}(x) = \begin{cases} (-\infty, 1], \text{ if } x = 1 \\ \{x\}, \text{ if } x \in (1, 2) \\ [2, +\infty), \text{ if } x = 2. \end{cases}$$

Note that A^{F_K} is maximal monotone. The solution of the equilibrium problem for F on K is 1. As expected, $0 \in A^{F_K}(1)$ while $0 \notin A^F(1)$. The following proposition shows that maximality of A^{F_K} follows from a qualification condition:

PROPOSITION 4.8 *Let $F : C \times C \to \mathbb{R}$ be maximal monotone and $K \subseteq C$ be a closed convex set such that $0 \in \operatorname{int}(K - D(A^F))$. Then F_K is maximal monotone and $A^{F_K}(x) = A^F(x) + N_K(x)$ for all $x \in K$, where*

$$N_K(x) = \{x^* \in X^* : \forall y \in K,\ \langle x^*, y - x \rangle \leq 0\}$$

is the normal cone to K at x.

Proof The subdifferential $\partial\delta_K$ of the indicator function δ_K of K is a maximal monotone operator and it is known that $\partial\delta_K(x) = N_K(x)$ for every $x \in K$, while $\partial\delta_K(x) = \emptyset$ for $x \in X \backslash K$. Since A^F is also maximal monotone, it follows from our assumptions that $A^F + \partial\delta_K$ is maximal monotone [10]. It is clear that $D(A^F + \partial\delta_K) \subseteq K$. For every $x \in D(A^F + \partial\delta_K)$ and $x^* \in (A^F + \partial\delta_K)(x)$ there exist $x_1^* \in A^F(x)$ and $x_2^* \in N_K(x)$ such that $x^* = x_1^* + x_2^*$. Thus for every $y \in K$,

$$F(x, y) \geq \langle x_1^*, y - x \rangle \geq \langle x_1^*, y - x \rangle + \langle x_2^*, y - x \rangle = \langle x^*, y - x \rangle .$$

Hence $(A^F + \partial\delta_K)(x) \subseteq A^{F_K}(x)$ for all $x \in X$. The proposition follows since $A^F + \partial\delta_K$ is maximal monotone. $\square$

5 Cyclically monotone bifunctions

We can define cyclic monotonicity as follows: A bifunction F is called cyclically monotone if

$$\forall x_1, x_2, \ldots, x_n \in C, \quad F(x_1, x_2) + F(x_2, x_3) + \ldots + F(x_n, x_{n+1}) \leq 0$$

where $x_{n+1} := x_1$. The following proposition provides a necessary and sufficient condition for a bifunction to be cyclically monotone.

PROPOSITION 5.1 *A bifunction $F : C \times C \to \mathbb{R}$ is cyclically monotone if, and only if, there exists a function $f : C \to \mathbb{R}$ such that*

$$\forall x, y \in C, \quad F(x, y) \leq f(y) - f(x). \tag{12}$$

Proof The "if" part is trivial. To show the "only if" part, we follow Rockafellar's proof for the representation of cyclically monotone operators via subdifferentials of convex functions [11], which we include for the sake of completeness. Assume that F is cyclically monotone. Choose any $x_0 \in C$ and define f on C by

$$f(x) = \sup\{F(x_0, x_1) + F(x_1, x_2) + \cdots + F(x_n, x)\} \tag{13}$$

where the supremum is taken over all families of elements $x_1, x_2, \ldots, x_n$ in C, for all $n \in \mathbb{N}$. Since

$$F(x_0, x_1) + F(x_1, x_2) + \cdots + F(x_n, x) + F(x, x_0) \leq 0$$

we deduce that $f(x) \leq -F(x, x_0)$ and in particular f is real valued. Also, for any $x, y \in C$ and $x_1, x_2, \ldots, x_n \in C$,

$$F(x_0, x_1) + F(x_1, x_2) + \cdots + F(x_n, x) + F(x, y) \leq f(y).$$

Taking the supremum over all families $x_1, x_2, \ldots, x_n$ we deduce that $f(x) + F(x, y) \leq f(y)$, i.e., (12) holds. □

Whenever F is also maximal monotone, more can be said on f:

PROPOSITION 5.2 *Suppose that* $\operatorname{int} C \neq \emptyset$ *and* $F : C \times C \to \mathbb{R}$ *is maximal monotone and cyclically monotone. Then the following statements are true:*

1) The sets $\overline{C}$ *and* $\operatorname{int} C$ *are convex, and equalities* $\overline{C} = \overline{D(A^F)}$ *and* $\operatorname{int} C = \operatorname{int} D(A^F)$ *hold; the function f in relation (12) is uniquely defined up to a constant on* $\operatorname{int} C$*, and is convex and continuous on* $\operatorname{int} C$*.*

2) If in addition $F(x,\cdot)$ is l.s.c. for every $x \in C$, then f is uniquely defined up to a constant, and convex and l.s.c. on C.

Proof 1) Our assumptions imply that A^F is maximal monotone and cyclically monotone. By a well-known theorem of Rockafellar [11], given $(x_0, x_0^*) \in \operatorname{gr} A^F$, the function $\phi : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\phi(x) = \sup_{\substack{n\in\mathbb{N},(x_i,x_i^*)\in\operatorname{gr} A^F\\ i=1,2,\dots n}} \{\langle x_0^*, x_1 - x_0\rangle + \langle x_1^*, x_2 - x_1\rangle + \cdots + \langle x_n^*, x - x_n\rangle\}$$

is a proper, l.s.c. and convex function such that $\partial\phi = A^F$. For every $x \in C$,

$$\begin{aligned}\phi(x) &= \sup_{\substack{n\in\mathbb{N},(x_i,x_i^*)\in\operatorname{gr} A^F\\ i=1,2,\dots n}} \{\langle x_0^*, x_1 - x_0\rangle + \langle x_1^*, x_2 - x_1\rangle + \cdots + \langle x_n^*, x - x_n\rangle\}\\ &\le \sup_{\substack{n\in\mathbb{N},x_i\in D(A^F)\\ i=1,2,\dots n}} \{F(x_0,x_1) + F(x_1,x_2) + \cdots + F(x_n,x)\} \le -F(x,x_0)\end{aligned}$$

since $F(x_0,x_1)+F(x_1,x_2)+\cdots+F(x_n,x)+F(x,x_0) \le 0$ by cyclic monotonicity. Hence ϕ is real on C so that $C \subseteq \operatorname{dom}(\phi)$. It follows that

$$\overline{C} \subseteq \overline{\operatorname{dom}(\phi)} = \overline{D(\partial\phi)} = \overline{D(A^F)} \subseteq \overline{C},$$
$$\operatorname{int} C \subseteq \operatorname{int}\operatorname{dom}(\phi) = \operatorname{int} D(\partial\phi) = \operatorname{int} D(A^F) \subseteq \operatorname{int} C$$

Consequently $\overline{C} = \overline{\operatorname{dom}(\phi)}$ and $\operatorname{int} C = \operatorname{int}\operatorname{dom}(\phi)$ are convex. Now let $f : C \to \mathbb{R}$ be any function such that (12) holds. Then for every $(x,x^*) \in \operatorname{gr} A^F$ and every $y \in C$ one has

$$f(y) - f(x) \ge \langle x^*, y - x\rangle\,.$$

This means that $\partial\phi \subseteq \partial f$ and by maximal monotonicity of $\partial\phi$, $\partial\phi = \partial f$.

For any $x, y \in C$ and $\lambda \in (0,1)$ with $z := (1-\lambda)x + \lambda y \in \operatorname{int} C$, choose $z^* \in A^F(z)$. Then from the inequalities

$$f(x) - f(z) \ge \langle z^*, x - z\rangle$$
$$f(y) - f(z) \ge \langle z^*, y - z\rangle$$

we deduce that

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y). \tag{14}$$

Hence f is convex on $\operatorname{int} C$. Also, f is l.s.c. on $\operatorname{int} C$ since $\partial f \neq \emptyset$ there. Since $\partial\phi = \partial f$, the functions ϕ and f differ by a constant on $\operatorname{int} C$.

2) Assume that F is l.s.c. and let f be a function satisfying (12). Then for every $x \in C$,

$$\liminf_{y\to x}\,(f(y) - f(x)) \geq \liminf_{y\to x} F(x,y) \geq F(x,x) = 0$$

thus f is l.s.c.. From part 1 of the proof we know that $\operatorname{int} C$ and $\overline{C}$ are convex. Adding a constant if necessary, we may assume that $f = \phi$ on $\operatorname{int} C$. For any $x \in C$, choose $y \in \operatorname{int} C$ and a sequence $x_n = (1-\lambda_n)x + \lambda_n y$, $n \in \mathbb{N}$, with $\lambda_n > 0$ and $\lambda_n \to 0$. Since $C \subseteq \operatorname{dom}(\phi)$ and $\operatorname{int} C = \operatorname{int}\operatorname{dom}(\phi)$, we have $x_n \in \operatorname{int}\operatorname{dom}(\phi) = \operatorname{int} C$. Using (14) which is valid whenever $(1-\lambda)x + \lambda y \in \operatorname{int} C$ and lower semicontinuity of f we obtain

$$\liminf_n f(x_n) \leq \liminf_n\,((1-\lambda_n)f(x) + \lambda_n f(y)) = f(x) \leq \liminf_n f(x_n).$$

Hence $f(x) = \liminf_n f(x_n)$. Using the same argument for ϕ, we deduce that $\phi(x) = \liminf_n \phi(x_n)$. Consequently,

$$f(x) = \liminf_n f(x_n) = \liminf_n \phi(x_n) = \phi(x).$$

Thus $f = \phi$ on C and this implies that f is convex. $\square$

The following example shows that a convex function such that (12) holds may not exist if $F(x,\cdot)$ is not l.s.c..

Example 5.3 Let $\phi : \mathbb{R}^2 \to \mathbb{R}\cup\{+\infty\}$ be the function with $\operatorname{dom}(\phi) = Q := [-1,1]\times[-1,1]$ defined by

$$\phi(a,b) = -\left(\sqrt{1-a^2} + \sqrt{1-b^2}\right).$$

Then ϕ is convex, differentiable on $\operatorname{int} Q$ and such that $D(\partial\phi) = \operatorname{int} Q$. Let further f be any function with $\operatorname{dom}(f) = Q$ and such that $f = \phi$ on $\operatorname{int} Q$, $f \geq \phi$ on ∂Q, and f is not convex (for instance, $f = \phi$ on $\mathbb{R}^2\backslash\{(1,0)\}$ and $f(1,0) = 1$). Define the cyclically monotone bifunction $F : Q\times Q \to \mathbb{R}$ by

$$F(x,y) = f(y) - f(x).$$

For every $x \in \operatorname{int} Q$ and $y \in Q$,

$$F(x,y) = f(y) - f(x) \geq \phi(y) - \phi(x) \geq \langle \nabla\phi(x), y-x\rangle.$$

Since $D(\partial\phi) = \operatorname{int} Q$, it follows that $\partial\phi(x) \subseteq A^F(x)$ for every $x \in \mathbb{R}^2$. By maximal monotonicity of $\partial\phi$ we obtain that $A^F = \partial\phi$ hence F is maximal monotone.

Now assume that there exists a convex function $f_1 : Q \to \mathbb{R}$ such that

$$\forall x, y \in Q,\ F(x, y) \leq f_1(y) - f_1(x).$$

Fix $y \in Q$. Since $f(y) - f(x) \leq f_1(y) - f_1(x)$ for all $x \in Q$, interchanging x and y we obtain $f(x) - f(y) \leq f_1(x) - f_1(y)$ and finally

$$\forall x \in Q,\ f(x) = f_1(x) - f_1(y) + f(y) = f_1(x) + k$$

where k is a constant. This means that f is convex, a contradiction.

Next example shows that cyclic monotonicity of A^F does not imply cyclic monotonicity of F, even if F is monotone, C is a convex subset of $\mathbb{R}$ and A^F is a subdifferential of a proper l.s.c. convex function.

Example 5.4 Let $C = (-1, 1]$ and $f(x) = \frac{1}{1-x^2}$ for $x \in (-1, 1)$. We define $F : C \times C \to \mathbb{R}$ by $F(x, y) = f(y) - f(x)$ whenever $x, y \in (-1, 1)$, $F(x, 1) = f'(x)(1-x)$ and $F(1, x) = -f'(x)(1-x)$ for $x \in (-1, 1)$, and $F(1, 1) = 0$. It is obvious that F is monotone. We compute A^F. For every $x \in (-1, 1)$ and any $y \in (-1, 1]$ it is clear from the definition that

$$F(x, y) \geq f'(x)(y - x).$$

Thus $\partial f(x) = \{f'(x)\} \subseteq A^F(x)$ for all $x \in D(\partial f)$. It follows by maximality of ∂f that $A^F = \partial f$ and in particular A^F is cyclically monotone. However, F is not cyclically monotone since

$$F(\frac{-1}{2}, \frac{1}{2}) + F(\frac{1}{2}, 1) + F(1, \frac{-1}{2}) = 0 + f'(\frac{1}{2})(1 - \frac{1}{2}) - f'(\frac{-1}{2})(1 + \frac{1}{2}) = 2f'(\frac{1}{2}) > 0.$$

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